SUBHARMONIC FUNCTIONS AND UNIFORM ALGEBRAS[1]

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ABSTRACT. Recently Aupetit and Wermer [J. Functional Anal. **28** (1978), 386–400] have shown conditions under which analytic structure exists in the spectrum space of a uniform algebra. Their work makes critical use of subharmonicity properties of certain classes of functions. In this paper, we develop a technique which offers an alternate and unified proof for sub-harmonicity of the functions in their paper assuming Basener's generalized Shilov boundary conjecture. Our technique uses the Oka-Wermer method applied to the n-fold tensor product of the algebra. We exhibit further applications of our main result including a special case which holds for all uniform algebras.

Introduction. Let A be a uniform algebra on a compact Hausdorff space X with spectrum M. We know that in many cases $M \smallsetminus X$ has a natural analytic structure. If the structure is one-dimensional and if $f \in A$, then locally either f is constant or f defines a covering map of a portion of $M \smallsetminus X$ onto a subset of the complex plane. This suggests that the mapping properties of functions in A should be used to exhibit the analytic structure in $M \smallsetminus X$, and indeed some deep results have been obtained along this line [1]. Recently, Aupetit and Wermer proved the following [2], which extended the earlier results by E. Bishop, J. Wermer, and R. F. Basener.

THEOREM 1. *Let W be a component of $C \smallsetminus f(X)$. Assume that there exists a subset G of W such that*

(1) *G has positive exterior capacity.*

(2) *For each $\lambda \in G$ the fiber over λ, $f^{-1}(\lambda) = \{ y \in M \mid f(y) = \lambda \}$, is denumerably infinite or finite.*

Then there exists an open dense subset of $f^{-1}(W)$ which can be given the structure of one-dimensional complex analytic space such that each $g \in A$ is analytic on this space.

The key tool used in the Aupetit-Wermer paper is the subharmonicity properties of certain classes of functions which arise in the following manner. For $f, g \in A$, suppose u is a set function from compact subsets of $g[f^{-1}(W)]$ into $\mathbf{R} \cup \{-\infty\}$. Some examples of such functions in the Aupetit-Wermer

paper are: $u_1(K) = \log\{\max_{z \in K}|z|\}$, $u_2(K) = \text{diameter}[K]$, $u_3(K) = \log\{\text{diameter}[K]\}$. For each such u_i associate a function $\mathcal{U}_i$ on W by

$$\mathcal{U}_i(x) = u_i\{g[f^{-1}(x)]\}, \qquad \lambda \in W.$$

It is the subharmonicity of the $\mathcal{U}_i$ that plays a large role in the Aupetit and Wermer approach. The subharmonicity of $\mathcal{U}_1$ is established by Wermer in an earlier paper [3] by combining an idea of Oka with results from the theory of uniform algebras; Aupetit and Wermer prove the subharmonicity of $\mathcal{U}_2$ by expressing u_2 in terms of u_1; they prove that $\mathcal{U}_3$ is subharmonic by applying potential theoretic results and the fact that $\mathcal{U}_2$ is subharmonic. In all of their proofs the characteristics of the individual functions are utilized in an important way.

In this paper we develop a technique to provide a single proof for the subharmonicity of all the $\mathcal{U}_i$ in the Aupetit-Wermer paper and a larger class of functions assuming Basener's generalized Shilov boundary conjecture. A detailed discussion of the conjecture will be given later.

Our technique, modeled in part after Wermer's proof for $\mathcal{U}_1$, is as follows: Consider the tensor product algebra $\otimes_n A$ of n copies of A. Define ψ on a component W of $\prod_1^n[f(M) \smallsetminus f(X)]$ by

$$\psi(\lambda_1, \ldots, \lambda_n) = \log\{\max|G(\phi_1, \ldots, \phi_n)|; (\phi_1, \ldots, \phi_n)$$
$$\in (f, \ldots, f)^{-1}(\lambda_1, \ldots, \lambda_n)\},$$

where $G \in \otimes_n A$. Using the Oka-Wermer method we show in Theorem 2 that ψ is plurisubharmonic.

Corollary 1 shows that if $n = 2$ then the restriction of ψ on the diagonal of W establishes the subharmonicity of all $\mathcal{U}_i$'s in the Aupetit-Wermer paper.

Corollary 2 shows that functions associated with transfinite diameter and capacity are subharmonic.

The generalized Shilov boundary condition, which Aupetit and Wermer did not require, is required in our proof because we have put their problem into the context of higher dimensional algebras and functions of several complex variables. It has been conjectured that this hypothesis can be removed, and recently R. F. Basener has verified that the generalized Shilov boundary condition holds for a large class of uniform algebras ([4], [5]). Our hypothesis is independent of conditions (1) and (2) in the Aupetit-Wermer theorem. Thus, our approach offers a potentially comprehensive and efficient technique for proving subharmonicity of functions associated with a uniform algebra.

The tensor product $\otimes_{i=1}^n A_i$ of uniform algebras A_i on X_i is the smallest closed subalgebra of $C(\prod_{i=1}^n X_i)$ generated by functions of the form $h_1(x_1) \cdots h_n(x_n)$, where $h_i \in A_i$ and $x_i \in X_i$. By $\otimes_n A$ we mean the tensor product of n copies of A.

Let $A^n = \{(f_1, \ldots, f_n)|f_1, \ldots, f_n \in A\}$, so that each $F = (f_1, \ldots, f_n) \in A_n$ maps M into $\mathbf{C}^n$. If K is a compact subset of M let $A_K = \{f \in C(K)|f \text{ is a}$

uniform limit on K of functions from A}. For $F \in A^n$, let $V(F) = \{x \in M \mid F(x) = 0\}$. $V(F)$ is A-convex, i.e. the maximal ideal space of $A_{V(F)}$ is $V(F)$.

DEFINITION. $\partial_n A = \text{cl}[\cup \{\partial_0 A_{V(F)} \mid F \in A^n\}]$. $\partial_n A$ is called the nth Shilov boundary of A. $\partial_0 A$ is the usual Shilov boundary of A. It has been conjectured that the following condition is true for arbitrary uniform algebras, A_j, $j = 1, \ldots, n$:

$$\partial_i \left[\bigotimes_{j=1}^n A_j \right] = \bigcup_{i_1 + \cdots + i_j = i} \prod_{j=1}^n \partial_{i_j} A_j.$$

R. F. Basener [5] has recently proved the condition to be true for $\bigotimes_{j=1}^n A_j$ where $A_j = P(X_j)$ and X_j is polynomial polyhedron. He has also shown that if A_1 is an arbitrary uniform algebra and A_2, any polydisc algebra, then $A_1 \otimes A_2$ satisfies (∗) for $i = 0, 1$. We are ready for our main theorem.

THEOREM 2. *Let X, A, and M be as before. Assume $\bigotimes_n A$ satisfies the condition (∗). For $f \in A$ let $\mathcal{F}$ be the map on $\prod_1^n M$ with $\mathcal{F}(\varphi_1, \ldots, \varphi_n) = (f(\varphi_1), \ldots, f(\varphi_n))$. Let W be a component of $\prod_1^n [f(M) \setminus f(X)]$. For $G \in \bigotimes_n A$ and $\lambda \in W$, put*

$$\psi(\lambda) = \log\{\max|G(\varphi)|: \varphi \in \mathcal{F}^{-1}(\lambda)\}.$$

Then ψ is plurisubharmonic on W.

PROOF. First we show ψ is upper semicontinuous on W. Fix $\lambda \in W$. Assume $\{\lambda^k\}_{k=1}^\infty \to \lambda$. Suppose $\psi(\lambda^k) \geqslant l > -\infty$ for all k. We claim $\psi(\lambda) \geqslant l$:

$$\psi(\lambda^k) = \max\{\log|G(\varphi^k)|: \varphi^k \in \mathcal{F}^{-1}(\lambda^k)\}.$$

Since $\mathcal{F}^{-1}(\lambda^k)$ is compact for each k, there exists $\varphi^k \in \prod_1^n M$, the maximal ideal space of $\bigotimes_n A$, such that $\varphi^k = \mathcal{F}^{-1}(\lambda^k)$ and $\log|G(\varphi^k)| \geqslant l$. By compactness of $\prod_1^n M$ we can find a subnet $\{\varphi^\alpha\}$ of $\{\varphi^k\}$ which converges to φ in $\prod_1^n M$.

$$\mathcal{F}(\varphi) = \lim_\alpha \mathcal{F}(\varphi^\alpha) = \lim_k \mathcal{F}(\varphi^k) = \lim_k \lambda^k = \lambda,$$
$$\log|G(\varphi)| = \lim_\alpha \log|G(\varphi^\alpha)| \geqslant l.$$

So, $\psi(\lambda) = \max\{\log|G(\varphi)|: \varphi \in \mathcal{F}^{-1}(\lambda)\} \geqslant l$. This shows that ψ is u.s.c. on W.

It remains to be shown that if T is an arbitrary linear transformation, $\psi \circ T$ is subharmonic on W in each variable. We will show ψ restricted to a given complex line, L, in W is subharmonic. Let

$$L = \bigcap_{j=1}^{n-1} \left\{ (\lambda_1, \ldots, \lambda_n) \in W \mid \sum_{i=1}^n \alpha_{ij} \lambda_i = \gamma_j \right\}; \qquad \alpha_{ij}, \gamma_j \in \mathbf{C}.$$

Let Δ be a closed disc contained in L and $\partial \Delta$ be the boundary of Δ in the topology of L. Choose a polynomial P such that $\psi|_L \leqslant \text{Re } P$ on $\partial \Delta$; i.e., $\psi|_L \cdot |e^{-P}| \leqslant 1$ on $\partial \Delta$.

Put $f_i(\varphi_1, \ldots, \varphi_n) = f(\varphi_i)$;

$$F^j = \sum_{i=1}^{n} \alpha_{ij} f_i - \gamma_j, \qquad j = 1, \ldots, n-1;$$

$$V = \left\{ \varphi \in \prod_1^n M \,\middle|\, F^j(\varphi) = 0, j = 1, \ldots, n-1 \right\};$$

$$\tilde{X} = f^{-1}[f(X)].$$

Let Q_i be the Cartesian product of $\tilde{X}$ with $n-1$ copies of M in which $\tilde{X}$ appears at the ith entry. Thus, $Q_1 = \tilde{X} \times M \times \cdots \times M$. From the hypothesis we have,

$$\partial_{n-1} \bigotimes_n A = \bigcup_{\substack{j_1 + \cdots + j_n = n-1;\\ j_1, j_2, \ldots, j_n > 0}} \left[\partial_{j_1} A \times \partial_{j_2} A \times \cdots \times \partial_{j_n} A \right] \subseteq \bigcup_{i=1}^{n} Q_i.$$

Let $\bigotimes_n A_V$ be the restriction algebra, $\{ g \in C(V) \mid g$ is a uniform limit on V of functions from $\bigotimes_n A \}$. Let $\partial_0 \bigotimes_n A_V$ to be the usual Shilov boundary of $\bigotimes_n A_V$. By the definition of generalized Shilov boundary, it follows that, $\partial_0 \bigotimes_n A_V \subseteq \partial_{n-1} \bigotimes_n A \subseteq \bigcup_{i=1}^n Q_i$.

Denote $N = \mathcal{F}^{-1}(\Delta)$. We claim $N \subseteq V \setminus \partial_0 \bigotimes_n A_V$. To prove the claim, we show $N \subseteq V \setminus \bigcup_{i=1}^n Q_i$. Suppose $\varphi \in N$. Then, $\mathcal{F}(\varphi) \in \mathring{\Delta} \subseteq L$. So

$$\sum_{i=1}^{n} \alpha_{ij} f_i(\varphi) = \gamma_j, \qquad j = 1, \ldots, n-1.$$

Hence, $\varphi \in V$. If $\varphi \in \bigcup_{i=1}^n Q_i$ then $\varphi_i \in \tilde{X}$ for some $1 \leqslant i \leqslant n$. But, $f(\varphi_i) \in f(M) \setminus f(X)$. So $\varphi_i \notin \tilde{X}$. Hence $N \subset V \setminus \bigcup_{i=1}^n Q_i$ as claimed.

Fix $\xi \in \mathring{\Delta}$ and choose $\theta = (\theta_1, \ldots, \theta_n) \in \mathcal{F}^{-1}(\xi)$ on which $|G|$ takes the maximum. By the local maximum modulus principle of $\bigotimes_n A_V$ applied to N,

$$|G(\theta)| \cdot \left| e^{-P(f(\theta_1), \ldots, f(\theta_n))} \right| \leqslant |G(\eta)| \cdot \left| e^{-P(f(\eta_1), \ldots, f(\eta_n))} \right|$$

for some $\eta \in \partial N$, where ∂N is the boundary of N relative to V. But, $\partial N \subseteq \mathcal{F}^{-1}(\partial \Delta)$. So

$$|G(\theta)| \cdot \left| e^{-P(f(\theta_1), \ldots, f(\theta_n))} \right| \leqslant 1.$$

Thus, we conclude, for arbitrary $\xi \in \mathring{\Delta}$,

$$\log \max\{ |G(\varphi)| : \varphi \in \mathcal{F}^{-1}(\xi) \} \leqslant \operatorname{Re} P(\xi).$$

We have shown that ψ is plurisubharmonic on W under the hypothesis (∗); however, we show in Theorem 3 that ψ is n-subharmonic without any hypothesis on the generalized Shilov boundary.

Recall that an upper semicontinuous function ψ on an open subset Ω of $\mathbf{C}^n$ is n-subharmonic if $-\infty \leqslant \psi < \infty$, $\psi \not\equiv -\infty$ on any component of Ω and ψ is subharmonic in each variable.

LEMMA 1. *Suppose u is upper semicontinuous on an open subset Ω of $\mathbf{C}^n$, and $-\infty \leqslant u < \infty$. Assume $u \not\equiv -\infty$ on any component of Ω. If u satisfies the condition below, then u is n-subharmonic:*

Given any polydisc $D \subseteq \Omega$ and a polynomial P on $\mathbf{C}^n$, if $u \leqslant \operatorname{Re} P$ on $\partial^0 D$, the distinguished boundary of D, then $u \leqslant \operatorname{Re} P$ on D.

THEOREM 3. *Notations as before.*

$$\psi(\lambda) = \log\{\max|G(\theta)|; \theta \in \mathcal{F}^{-1}(\lambda)\}$$

is n-subharmonic on W.

PROOF. For simplicity of notation we prove the theorem for $n = 2$. The same argument holds for the general case. We have shown in Theorem 2 that ψ is u.s.c. on W. It suffices to show that ψ satisfies the condition of Lemma 1.

Let $D = D_1 \times D_2$ be a bidisc in W, with $\partial^0 D$ its distinguished boundary. Choose a polynomial P such that $\psi \leqslant \operatorname{Re} P$ on $\partial^0 D$. That is, suppose for each $(y_1, y_2) \in \partial^0 D$, if $(\varphi_1, \varphi_2) \in \mathcal{F}^{-1}(y_1, y_2)$. Then

$$|G(\varphi_1, \varphi_2)| \cdot |e^{-P(f(\varphi_1), f(\varphi_2))}| \leqslant 1. \tag{1}$$

Let (x_1, x_2) be an arbitrary point in $\mathring{D}$. By compactness of $\mathcal{F}^{-1}(x_1, x_2)$, there is a point $(\theta_1, \theta_2) \in \mathcal{F}^{-1}(x_1, x_2)$ on which $|G|$ takes the maximum value. We will show

$$|G(\theta_1, \theta_2)| \cdot |e^{-P(x_1, x_2)}| \leqslant 1.$$

Fix θ_1. Define $H(\theta) = G(\theta_1, \theta) \cdot e^{-P(f(\theta_1), f(\theta))}$ for $\theta \in M$. It is easy to show that $H \in A$. Recall that $(\theta_1, \theta_2) \in \mathcal{F}^{-1}(x_1, x_2)$ where $(x_1, x_2) \in \mathring{D}$. Hence, $\theta_2 \in f^{-1}(\mathring{D}_2)$. Applying the local maximum modulus principle for A to $f^{-1}(\mathring{D}_2)$, we obtain $|H(\theta_2)| \leqslant |H(\theta_2')|$ for some $\theta_2' \in \partial[f^{-1}(\mathring{D}_2)]$. Note $\partial_0 A \cap f^{-1}(\mathring{D}_2) = \varnothing$. We have,

$$|G(\theta_1, \theta_2)| \cdot |e^{-P(f(\theta_1), f(\theta_2))}| \leqslant |G(\theta_1, \theta_2')| \cdot |e^{-P(f(\theta_1), f(\theta_2'))}|. \tag{2}$$

Now fix θ_2'. Let $V = \{(\theta, \theta_2') | \theta \in f^{-1}(\mathring{D}_1)\}$. We claim that $\partial_0(A \otimes A_V) \subset \mathcal{F}^{-1}(\partial^0 D)$.

PROOF OF THE CLAIM. For each $h \in A \otimes A$, define $\tilde{h}(x) = h(x, \theta_2')$. Then $\tilde{h} \in A$. Apply the local maximum modulus principle to $f^{-1}(\mathring{D}_1)$ and obtain

$$\|\tilde{h}\|_{\overline{f^{-1}(\mathring{D}_1)}} = \|\tilde{h}\|_{\partial\left[\overline{f^{-1}(\mathring{D}_1)}\right]}.$$

So, for each $h \in A \otimes A$ and $(x, \theta_2') \in V$, we can find some

$$(x', \theta_2') \in \partial\left[f^{-1}(\mathring{D}_1)\right] \times f^{-1}(\partial D_2) \subseteq f^{-1}(\partial D_1) \times f^{-1}(\partial D_2) = \mathcal{F}^{-1}(\partial^0 D)$$

such that $|h(x, \theta_2')| \leqslant |h(x', \theta_2')|$. This shows that $\mathcal{F}^{-1}(\partial^0 D) \cap V$ is a closed boundary for $A \otimes A_V$. Hence $\partial_0(A \otimes A_V) \subseteq \mathcal{F}^{-1}(\partial^0 D)$. Hence there exists $(\xi, \theta_2') \in \partial_0(A \otimes A_V)$ such that

$$|G(\theta_1, \theta_2')| \cdot |e^{-P(f(\theta_1), f(\theta_2'))}| \leqslant |G(\xi, \theta_2')| \cdot |e^{-P(f(\xi), f(\theta_2'))}|. \tag{3}$$

By (2), (3), and (1),

$$|G(\theta_1, \theta_2)| \cdot |e^{-P(x_1, x_2)}| \leqslant |G(\xi, \theta_2')| \cdot |e^{-P(y_1, y_2)}| \leqslant 1.$$

This concludes the proof of Theorem 3.

In the remainder of this paper, we shall discuss applications of Theorem 2. In [2], B. Aupetit and J. Wermer developed sufficient conditions for the existence of analytic structure in the maximal ideal space of a uniform algebra, improving on the earlier results by R. Basener [6] and E. Bishop [7]. Aupetit and Wermer's proof depends critically on subharmonicity properties exhibited by certain classes of functions associated with the uniform algebra. Presently, we shall focus our attention on these functions.

NOTATION. Fix $g \in A$. For $\lambda \in W$, a component of $\mathbf{C} \setminus f(X)$, denote:

$$K_g(\lambda) = g[f^{-1}(\lambda)] = \{g(y): y \in f^{-1}(\lambda)\} \subset \mathbf{C};$$

$$Z_g(\lambda) = \max\{|Z|: Z \in K_g(\lambda)\};$$

$$r_g(\lambda) = \max\{1/|Z|: Z \in K_g(\lambda)\};$$

$$d_g(\lambda) = \text{diameter of } K_g(\lambda).$$

Aupetit and Wermer constructed the following functions and have shown that each of them is subharmonic on W for all f and g in A:

Function 1: $\lambda \to Z_g(\lambda)$;
Function 2: $\lambda \to \log Z_g(\lambda)$;
Function 3: $\lambda \to d_g(\lambda)$;
Function 4: $\lambda \to |\exp P(x)| \cdot d_g(\lambda)$, where P is a polynomial;
Function 5: $\lambda \to \log d_g(x)$.

They have also shown that for fixed $\lambda_0 \in W$, and a complex number α with $\alpha \notin \hat{K}_g(\lambda_0)$, the polynomial hull of $K_g(\lambda_0)$,

Function 6: $\lambda \to r_{g-\alpha}(\lambda)$ is subharmonic in a neighborhood of λ_0.

Subharmonicity of certain classes of functions associated with $K_g(\lambda)$ including the ones listed above follows from Theorem 2.

COROLLARY 1 TO THEOREM 2. *Suppose $A \otimes A$ satisfies the hypothesis $(*)$. Then Functions (1)–(6) listed above are subharmonic.*

PROOF. If we let $n = 2$, Theorem 2 states that

$$\psi(\lambda_1, \lambda_2) = \log \max\{|G(\theta_1, \theta_2)| \,|(\theta_1, \theta_2) \in \mathscr{F}^{-1}(\lambda_1, \lambda_2)\}$$

is plurisubharmonic. Hence ψ is subharmonic in each variable. Let $G(\theta_1, \theta_2) = g(\theta_1)$. Then $\log \max\{|g(\theta_1)|: \theta_1 \in f^{-1}(\lambda)\}$ is subharmonic. This shows that Function 2 is subharmonic. If $\log|h|$ is subharmonic so is $|h|$. Hence Function 1 is subharmonic. ψ is subharmonic on the diagonal of each component of W. By setting $G(\theta_1, \theta_2) = g(\theta_1) - g(\theta_2)$ we have, $\log \max\{|g(\theta_1) - g(\theta_2)|: (\theta_1, \theta_2) \in \mathscr{F}^{-1}(\lambda, \lambda)\}$ is subharmonic. This shows that Functions 3 and 5 are subharmonic. $\log(|\exp P(\lambda)| \cdot d_g(\lambda)) = \text{Re } P(\lambda) + \log d_g(\lambda)$. So, Function 4 is subharmonic. Subharmonicity of Function 6 is proved in [2] via subharmonicity of $\log Z_g$, Function 2. □

COROLLARY 2 TO THEOREM 2. *Suppose $\bigotimes_n A$ satisfies the condition (∗). Then $\lambda \to \text{Capacity}[K_g(\lambda)]$ is subharmonic on each component of $f(M) \setminus f(X)$.*

A theorem by Szegö relates capacity to a geometric quantity called transfinite diameter [8]. Diameter of order n of a closed bounded set, E, denoted by $d_n(E)$ is

$$d_n(E) = \max\left\{ \prod |z_i - z_j|^{2/n(n-1)}, z_i, z_j \in E, 1 \leqslant i < j \leqslant n \right\}.$$

If $n = 2$, $d_n(E)$ is the ordinary diameter of E. It is straightforward to show that $d_n \leqslant d_{n-1} \leqslant \cdots \leqslant \text{diameter } (E) < \infty$. We set $d = \lim_n d_n$ and call it the transfinite diameter.

THEOREM 4 (SZEGÖ). *The capacity of a closed bounded set is equal to its transfinite diameter.*

We are now ready to prove Corollary 2.

PROOF. Since the monotone decreasing limit of subharmonic functions is subharmonic, we need only to prove that for all $n \geqslant 2$, $\lambda \to d_n[K_g(\lambda)]$ is subharmonic. We shall use the notation $d_n(\lambda)$ to mean $d_n[K_g(\lambda)]$. If $g \in A$ and $\varphi = (\varphi_1, \ldots, \varphi_n) \in \Pi_1^n M$, denote $g_i(\varphi) = g(\varphi_i)$, $i = 1, \ldots, n$. $g_i \in \bigotimes_n A$ for each i. Hence, $\Pi_{1 \leqslant i < j \leqslant n}(g_i - g_j) \in \bigotimes_n A$. For $(\lambda_1, \ldots, \lambda_n) \in \Pi_1^n[f(M) \setminus f(X)]$ put

$$V_n(\lambda_1, \ldots, \lambda_n) = \max\left\{ \prod_{1 \leqslant i < j \leqslant n} |g_i(\varphi) - g_j(\varphi)| : \varphi \in \mathcal{F}^{-1}(\lambda_1, \ldots, \lambda_n) \right\}.$$

Then $d_n(\lambda) = V_n(\lambda, \ldots, \lambda)^{2/n(n-2)}$, i.e., d_n is $V_n^{2/n(n-2)}$ restricted to the diagonal. By Theorem 2, $\log V_n$ is plurisubharmonic on W, hence subharmonic on the diagonal of W. This establishes subharmonicity of d_n, since

$$\log d_n = \frac{2}{n(n-1)} \log V_n \quad \text{and} \quad \frac{2}{n(n-1)} > 0.$$

BIBLIOGRAPHY

1. J. Wermer, *Banach algebras and several complex variables*, Springer-Verlag, Berlin and New York, 1976, pp. 64–70, pp. 137–142.

2. B. Aupetit and J. Wermer, *Capacity and uniform algebras*, J. Functional Anal. **28** (1978), 386–400.

3. J. Wermer, *Subharmonicity and hulls*, Pacific J. Math. **8** (1975), 283–290.

4. R. F. Basener, *A generalized Shilov boundary and analytic structure*, Proc. Amer. Math. Soc. **47** (1975), 98–104.

5. _____, *Boundaries for product algebras* (preprint).

6. _____, *A condition for analytic structure*, Proc. Amer. Math. Soc. **36** (1972), 156–160.

7. E. Bishop, *Holomorphic completions, analytic continuation, and the interpolation of seminorms*, Ann. of Math. (2) **78** (1963), 468–500.

8. G. M. Goluzin, *Geometric theory of functions of a complex variable*, Transl. Math. Monographs, vol. 26, Amer. Math. Soc., Providence, R. I., 1969.

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