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PERMUTATION GROUPS WITH PROJECTIVE UNITARY SUBCONSTITUENTS

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ABSTRACT. Let Γ be a finite directed graph with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$ and let G be a subgroup of aut(Γ) which we assume to act transitively on both $V(\Gamma)$ and $E(\Gamma)$. Suppose that for some prime power q, the stabilizer $G(x)$ of a vertex x induces on both $\{y|(x, y) \in E(\Gamma)\}$ and $\{w|(w, x) \in E(\Gamma)\}$ a group lying between $PSU(3, q^2)$ and $P\Gamma U(3, q^2)$. It is shown that if G acts primitively on $V(\Gamma)$, then for each edge (x, y), the subgroup of $G(x)$ fixing every vertex in $\{w|(x, w)$ or $(y, w) \in E(\Gamma)\}$ is trivial.

Let Γ be a directed graph with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$ and let G be a subgroup of aut(Γ) which we assume to act transitively on both $V(\Gamma)$ and $E(\Gamma)$. Let $x \in V(\Gamma)$ be arbitrary. We denote by $\Gamma(x)$ the set of $y \in V(\Gamma)$ such that $(x, y) \in E(\Gamma)$ and by $\Gamma'(x)$ the set of $w \in V(\Gamma)$ such that $(w, x) \in E(\Gamma)$. Since G acts transitively on $E(\Gamma)$, the stabilizer $G(x)$ of x in G acts transitively on both $\Gamma(x)$ and $\Gamma'(x)$ and either $\Gamma(x) \cap \Gamma'(x) = \varnothing$ or $\Gamma(x) = \Gamma'(x)$. In the latter case, we may identify Γ with the undirected graph with vertex set $V(\Gamma)$ and edge set $\{\{x, y\}|(x, y) \in E(\Gamma)\}$ and will, in fact, simply say that Γ itself is undirected. Let $G_1(x) = \{a \in G(x)|a \in G(y)$ for all $y \in \Gamma(x)\}$ and $G_1'(x) = \{a \in G(x)|a \in G(w)$ for all $w \in \Gamma'(x)\}$. For $u \in \Gamma(x)$ or $\Gamma'(x)$, we set $G(x, u) = G(x) \cap G(u)$ and $G_1(x, u) = G_1(x) \cap G_1(u)$. Γ is called connected if for every two vertices u and v, there is a sequence $(x_0, x_1, \ldots, x_s)$ of vertices such that $x_0 = u$, $x_s = v$ and $x_{i-1} \in \Gamma(x_i) \cup \Gamma'(x_i)$ for $1 < i < s$. The following observation is easily verified.

LEMMA 1. *If Γ is connected and undirected and $\{x, y\} \in E(\Gamma)$, then $\langle G(x), G(y)\rangle$ acts transitively on $E(\Gamma)$.*

Note that this statement does not hold if we do not assume Γ to be undirected. Consider, for instance, the graph Γ with $V(\Gamma) = \mathbf{Z}_k \times M$, M an arbitrary nonempty set and $k > 3$, and $E(\Gamma) = \{((i, x), (j, y))|i - j \equiv 1 \pmod k\}$ and $G = \text{aut}(\Gamma)$.

If G is an arbitrary transitive permutation group on a set Ω and Δ an orbit of G on $\Omega \times \Omega$ (called an orbital of G), then G can be considered as a vertex- and edge-transitive subgroup of aut(Γ_Δ) where Γ_Δ is the graph with vertex set Ω and edge set Δ. According to [9, (4.4)], G is primitive on Ω if and only if, for each nondiagonal orbital Δ, Γ_Δ is connected.

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In [2] and [3], Dempwolff proved the following result.

THEOREM 2. *Let q be a prime power, $q > 2$. Let Γ be a finite graph and $(x, y) \in E(\Gamma)$ arbitrary. Suppose* $\mathrm{aut}(\Gamma)$ *contains a subgroup G acting transitively on both $V(\Gamma)$ and $E(\Gamma)$ such that both $G(x)^{\Gamma(x)}$ (i.e., the permutation group induced by $G(x)$ on $\Gamma(x)$) and $G(x)^{\Gamma'(x)}$ are isomorphic to $PSU(3, q^2)$ in its usual 2-transitive representation of degree $q^3 + 1$. Suppose further that G is primitive on $V(\Gamma)$ or that Γ is undirected and connected. Then $G_1(x, y) = 1$.*

Actually, Dempwolff simply required G to be primitive on $V(\Gamma)$. This hypothesis is used, however, only in the following way.

Suppose H is a subgroup of $G(x, y)$ normalized by both $G(x)$ and $G(y)$. Since G is primitive and not regular on $V(\Gamma)$, $G = \langle G(x), G(y)\rangle$. Hence $H \triangleleft G$. Since H fixes the edge (x, y), H fixes every edge. Therefore $H = 1$.

In light of Lemma 1, this same conclusion (i.e., $H = 1$) holds if, instead of assuming $G^{V(\Gamma)}$ to be primitive, we assume Γ to be undirected and connected.

In this paper, we prove the following improved version of Theorem 2.

THEOREM 3. *Let q be an arbitrary prime power. Let Γ be a finite graph and $(x, y) \in E(\Gamma)$ arbitrary. Suppose* $\mathrm{aut}(\Gamma)$ *contains a subgroup G acting transitively on both $V(\Gamma)$ and $E(\Gamma)$ such that $PSU(3, q^2) \underset{\sim}{\leqslant} G(x)^{\Gamma(x)} \underset{\sim}{\leqslant} P\Gamma U(3, q^2)$ and $PSU(3, q^2) \underset{\sim}{\leqslant} G(x)^{\Gamma'(x)} \underset{\sim}{\leqslant} P\Gamma U(3, q^2)$. Suppose that G is primitive on $V(\Gamma)$ or that Γ is undirected and connected. Then $G_1(x, y) = 1$.*

Our proof of Theorem 3, although similar to the proof of Theorem 2 given in [2] and [3], is more elementary in that we avoid having to prove [2, (2.3)] and [3, (2.5)]. Moreover, our proof remains valid with only the most minor changes when we replace $PSU(3, q^2)$ by $Sz(q)$ or ${}^2G_2(q)$ (and $P\Gamma U(3, q^2)$ by the corresponding automorphism group) in the statement of Theorem 3 once it is shown that $Sz(q)$ and ${}^2G_2(q)$ have a certain elementary property. For $Sz(q)$, this property is easily checked; for ${}^2G_2(q)$, verification is more difficult. See the remarks at the end of our proof of Theorem 3. Of course, these versions of Theorem 3 were already proved in [1, (2.6) and (3.5)], but only by using the very deep results [4, Corollary 10] in the case $Sz(q)$ and [5, (1.4)] in the case ${}^2G_2(q)$.

We begin the proof of Theorem 3 by gathering the properties of $PSU(3, q^2)$ which will be needed. The reader unacquainted with these groups is referred to [7] (in particular, pp. 242–244). Let Π be the desarguesian plane $PG(2, q^2)$ over the field $GF(q^2)$, δ the unitary polarity of Π corresponding to a nondegenerate hermitian form on the underlying vector space and X the set of absolute points (i.e., those x in Π incident with x^δ). $PSU(3, q^2)$ is the subgroup of $PSL(3, q^2)$ consisting of those elements which commute with δ. Let $H = PSU(3, q^2)$. H acts 2-transitively on X and $|X| = q^3 + 1$. Let $x \in X$ be arbitrary. Let $q = p^n$, p prime. Then $O_pH(x)$ (i.e., the largest normal p-subgroup of the stabilizer $H(x)$) acts regularly on $X - \{x\}$ and its center $ZO_pH(x)$, which is of order q, consists of precisely those elements of $H(x)$ fixing all the q^2 nonabsolute lines (i.e., those lines L of Π not incident with L^δ) passing through x. It is straightforward to check by calculating with the elements denoted by $Q(a, b)$ and T in [7, pp. 243–244] that if

x_1, x_2 and $x_3 \in X$ are noncollinear, then $\langle ZO_pH(x_i)|1 \leqslant i \leqslant 3\rangle$ acts transitively on X and hence $\langle ZO_pH(x_i)|1 \leqslant i \leqslant 3\rangle = \langle ZO_pH(x)|x \in X\rangle$. If $q > 2$, we have $\langle ZO_pH(x)|x \in X\rangle = H$ since H is simple; when $q = 2$, $\langle ZO_pH(x)|x \in X\rangle = H'$.

LEMMA 4. *Let P_1 be a nontrivial subgroup of $ZO_pH(x)$ for some $x \in X$ and let $\mathcal{C}$ be the set of subgroups conjugate to P_1 in H. If $|P_1| > 2$ or $q = 2$, there exist P_2 and $P_3 \in \mathcal{C}$ such that $\langle P_1, P_2, P_3\rangle = H'$ (where $H = H'$ if $q > 2$). If $|P_1| = 2$ and $q > 2$, there exist P_2, P_3 and $P_4 \in \mathcal{C}$ such that $\langle P_1, P_2, P_3, P_4\rangle = H$.*

PROOF. Let Y be the set of absolute points on some nonabsolute line through x. H_Y induces $PGL(2, q)$ on Y. If q is even, there exists, according to [7, (8.8.27)], a subgroup $P_2 \in \mathcal{C}$ such that $\langle P_1, P_2\rangle^Y$ contains a dihedral group of order $2(q + 1)$ which is maximal in $H_Y^Y \cong PSL(2, q)$. If $|P_1| > 2$ or $q = 2$, we have $\langle P_1, P_2\rangle^Y \cong PSL(2, q)$. If $|P_1| = 2$ but $q > 2$, there is a subgroup $P_4 \in \mathcal{C}$, $P_4 \leqslant H(x)$, distinct from P_1; we have $\langle P_1, P_2, P_4\rangle^Y \cong PSL(2, q)$. If q is odd, it follows from [6, (2.8.4)] if $q \neq 9$ and from $PSL(2, 9) \cong A_6$ when $q = 9$ that there exists a $P_2 \in \mathcal{C}$ such that $\langle P_1, P_2\rangle^Y \cong PSL(2, q)$ since any two p-elements of $PSL(2, q)$ are conjugate in $PGL(2, q)$ and A_6 can be generated by two of its 3-elements.

Now suppose that for some subgroup A of H_Y, $A^Y \cong PSL(2, q)$, q even or odd. Then $ZO_pH(y) \leqslant A$ for every $y \in Y$. If P_3 is any subgroup in $\mathcal{C}$ whose absolute fixed point does not lie on Y, then $g(Y) \neq Y$ for every nontrivial element g in P_3 and so $\langle A, P_3\rangle$ contains $ZO_pH(z)$ for at least three noncollinear points $z \in X$. $\square$

In the proof of Theorem 3 we will need to deal with an arbitrary elementary abelian subgroup of $O_pH(x)$. If $p = 2$, every such subgroup is contained in $ZO_pH(x)$. For odd p, this is not so.

LEMMA 5. *Let $p \neq 2$, let P_1 be a nontrivial elementary abelian subgroup of $O_pH(x)$ for some $x \in X$ and let $\mathcal{C}$ be the set of subgroups conjugate to P_1 in H. Then $|P_1| \leqslant q^2$. If $P_1 \cap ZO_pH(x) = 1$ then $|P_1| \leqslant q$ and there exist P_2, P_3 and $P_4 \in \mathcal{C}$ such that $\langle P_1, P_2, P_3, P_4\rangle = H$. If $|P_1| > q$ then there exists a single subgroup P_2 such that $\langle P_1, P_2\rangle = H$.*

PROOF. It is easily checked that $|P_1| \leqslant q^2$ and $|P_1| \leqslant q$ if $P_1 \cap ZO_pH(x) = 1$. Suppose first that $|P_1| > q$ so that $P_1 \cap ZO_pH(x) \neq 1$. Given any nonabsolute line L through x, we may choose $P_2 \in \mathcal{C}$ such that $\langle P_1, P_2\rangle$ induces $PSL(2, q)$ on Y, where $Y = X \cap L$. Since $|P_1| > q$, $P_1 \nleqslant ZO_pH(x)$ and so P_1 contains elements which do not map Y to itself. Hence $\langle P_1, P_2\rangle$ contains $ZO_pH(z)$ for at least three noncollinear points $z \in X$. Thus $\langle P_1, P_2\rangle = H$. Now suppose $P_1 \cap ZO_pH(x) = 1$. Choose any nontrivial element $a \in P_1$. There exists an homology $h \in H(x)$ such that a and a^h do not commute. Let $P_2 = P_1^h$. Then $1 \neq [a, a^h] \in (O_pH(x))' = ZO_pH(x)$; thus $\langle P_1, P_2\rangle \cap ZO_pH(x) \neq 1$. It follows now just as in the previous case that there exists a subgroup P conjugate to $\langle P_1, P_2\rangle$ in H such that $\langle P_1, P_2, P\rangle = H$. $\square$

$P\Gamma U(3, q^2)$ is the subgroup of $P\Gamma L(3, q^2)$ consisting of those elements which commute with the polarity δ. Let $K = P\Gamma U(3, q^2)$. Then $K \cong \mathrm{aut}(H)$ and for each $x \in X$, $O_pH(x) = O_pK(x)$. If $H \leqslant L \leqslant K$, then p is the only local prime of $L(x)$, i.e., the only prime such that $O_pL(x) \neq 1$.

We now begin the actual proof of Theorem 3. Suppose Γ and G fulfill the hypotheses. According to [**8**, (4.9)] if $q = 2$ and [**8**, (3.5), condition (1)] if $q > 2$, $G_1(x) = G_1'(x)$. Suppose that $G_1(x, y) \neq 1$. Recalling our previous remarks, we note that [**8**, (4.11)] continues to hold when $G^{V(\Gamma)}$ is imprimitive if Γ is assumed to be undirected and connected. Thus $G_1(x, y)$ is a p-group since p is the only local prime of $G(x, y)^{\Gamma(x)}$.

Let $u \in \Gamma(x)$ or $\Gamma'(x)$ be arbitrary. We claim that $ZO_pG(x, u)$ is contained in $ZO_pG(x)$. Suppose the contrary. Since $O_p(G(x)^{\Gamma(x)}) = 1$, $O_pG(x) = O_pG_1(x) = O_pG(x, u) \cap G_1(x)$. It follows that $ZO_pG(x, u) \nleqslant G_1(x)$. Since $G_1(x) = G_1'(x)$, we can find $a \in ZO_pG(x, u)$, $z \in \Gamma(x)$ and $w \in \Gamma'(x)$ such that $a \notin G(z)$ and $a \notin G(w)$. Since $G_1(x, z) \trianglelefteq G_1(x)$, $G_1(x, z) \leqslant O_pG_1(x) \leqslant O_pG(x, u)$ and hence $G_1(x, z) = aG_1(x, z)a^{-1} = G_1(x, a(z))$. Therefore $G_1(x, z) \trianglelefteq \langle G(x, z), G(x, a(z))\rangle$. Since $G(x)^{\Gamma(x)}$ is primitive but not regular and $a(z) \neq z$, $\langle G(x, z), G(x, a(z))\rangle = G(x)$. Thus $G_1(x, z) \trianglelefteq G(x)$. Similarly, $G_1(w, x) \trianglelefteq G(x)$. If $b \in G$ is an element mapping (w, x) to (x, z) then $G_1(x, z) = bG_1(w, x)b^{-1} \trianglelefteq bG(x)b^{-1} = G(z)$. It follows that $G_1(x, z) \trianglelefteq \langle G(x), G(z)\rangle$ although $G_1(x, z) \neq 1$, a contradiction. Thus $ZO_pG(x, u) \leqslant ZO_pG(x)$ as claimed.

Let Ω_1 be the functor which assigns to a p-group the subgroup generated by its elements of order p. Let $V = \langle \Omega_1 ZO_pG(x, u) | u \in \Gamma(x) \cup \Gamma'(x)\rangle$. By the previous paragraph, $V \leqslant \Omega_1 ZO_pG(x)$. Let $C(V) = C_{G(x)}(V)$, the centralizer of V in $G(x)$, and suppose that $C(V) \nleqslant G_1(x)$. Since $G_1(x) = G_1'(x)$, we can find $a \in C(V)$, $z \in \Gamma(x)$ and $w \in \Gamma'(x)$ such that $a \notin G(z)$ and $a \notin G(w)$. Since $\Omega_1 ZO_pG(x, z) \leqslant V$, $\Omega_1 ZO_pG(x, z) = a\Omega_1 ZO_pG(x, z)a^{-1} = \Omega_1 ZO_pG(x, a(z))$ and thus $\Omega_1 ZO_pG(x, z) \trianglelefteq \langle G(x, z), G(x, a(z))\rangle = G(x)$. Similarly, $\Omega_1 ZO_pG(w, x) \trianglelefteq G(x)$. Conjugating $\Omega_1 ZO_pG(w, x)$ by an element mapping (w, x) to (x, z), we conclude that $\Omega_1 ZO_pG(x, z) \trianglelefteq G(z)$. Thus $\Omega_1 ZO_pG(x, z) \trianglelefteq \langle G(x), G(z)\rangle$ and so $\Omega_1 ZO_pG(x, z) = 1$ although $1 \neq G_1(x, z) \leqslant O_pG(x, z)$, a contradiction. It follows that $C(V) \leqslant G_1(x)$.

Let m denote the functor which assigns to a p-group the maximal order of an elementary abelian subgroup and J_1 the functor which assigns to a p-group the subgroup generated by all its elementary abelian subgroups of this order. Suppose $J_1O_pG(x, y) \leqslant G_1(x)$. Then $J_1O_pG(x, y) \leqslant O_pG(x, y) \cap G_1(x) = O_pG(x)$ and hence $mO_pG(x, y) = mO_pG(x)$ and $J_1O_pG(x, y) = J_1O_pG(x)$. Let $w \in \Gamma'(x)$ be arbitrary. Since G acts transitively on $K(\Gamma)$, $mO_pG(w, x) = mO_pG(x, y) = mO_pG(x)$. Since $O_pG(x) \leqslant O_pG(w, x)$, $J_1O_pG(x) \leqslant J_1O_pG(w, x)$. Since $|J_1O_pG(x)| = |J_1O_pG(x, y)| = |J_1O_pG(w, x)|$, $J_1O_pG(x) = J_1O_pG(w, x)$. Conjugating $J_1O_pG(w, x)$ by an element mapping (w, x) to (x, y), we see that $J_1O_pG(x, y) \trianglelefteq G(y)$. Thus $J_1O_pG(x, y) \trianglelefteq \langle G(x), G(y)\rangle$ and so $J_1O_pG(x, y) = 1$ although $O_pG(x, y) \neq 1$, a contradiction. It follows that $J_1O_pG(x, y) \nleqslant G_1(x)$.

Choose P_1 among those elementary abelian subgroups of $O_pG(x, y)$ of order $mO_pG(x, y)$ not contained in $G_1(x)$ and let $P_0 = P_1 \cap G_1(x)$. Since $V \leqslant \Omega_1 ZO_pG(x)$ and $P_0 \leqslant O_pG(x, y) \cap G_1(x) = O_pG(x)$, P_0V is elementary abelian. Hence $|P_0V| \leqslant |P_1|$. Since P_1 is abelian, $P_0 \cap V \leqslant C_V(P_1)$. Therefore $|P_1/P_0| \geqslant |P_0V/P_0| = |V/P_0 \cap V| \geqslant |V/C_V(P_1)|$. P_1/P_0 is isomorphic to a nontrivial elementary abelian p-subgroup of $O_p(G(x, y)^{\Gamma(x)})$. Let $|P_1/P_0| = p^m$. Let t be an integer such that there exist subgroups $P_2, \ldots, P_t$ conjugate to P_1 in $G(x)$ such that, with $A = \langle P_1, \ldots, P_t \rangle$, $A^{\Gamma(x)} \cong PSU(3, q^2)'$. Since P_i is conjugate to P_1, $|V/C_V(P_i)| = |V/C_V(P_1)|$ for $2 \leq i \leq t$; hence $|V/C_V(A)| \leq |V/C_V(P_1)|^t \leq |P_1/P_0|^t = p^{mt}$. Let $W = V/C_V(A)$ and $D = C_A(W)$. D is normal in A. If $D \nleq G_1(x)$, D contains elements of order prime to p not in $G_1(x)$. By [**6**, (5.3.2)], these elements lie in $C(V)$. This contradicts $C(V) \leq G_1(x)$. Hence $D \leq G_1(x)$. Since A/D is faithfully represented on W, we have $A/D \lesssim GL(W)$ and thus $q^3 + 1 \mid |A^{\Gamma(x)}| \mid |A/D| \mid |GL(W)|$ and so $q^3 + 1$ divides $(p^{mt} - 1)(p^{mt-1} - 1) \cdots (p - 1)$.

Since p^m is the order of a group isomorphic to an elementary abelian p-subgroup of $PSU(3, q^2)$, $p^m \leq q$ if q is even and $p^m \leq q^2$ if q is odd. If $q = 2$ (and thus $p^m = 2$), then according to Lemma 4 we can take $t = 3$. This implies that $2^3 + 1$ divides $(2^3 - 1)(2^2 - 1)$ which is not true. Thus $q > 2$. By [**11**, p. 283], there exists a prime π dividing $q^6 - 1$ but not $p^v - 1$ for any $v < 6n$ where $q = p^n$. In particular, π divides $q^3 + 1 = (q^6 - 1)/(q^3 - 1)$. Hence π divides $(p^{mt} - 1)(p^{mt-1} - 1) \cdots (p - 1)$ and so $6n \leq mt$. If $p^m > q$ then according to Lemma 5 we can take $t = 2$ and so $6n \leq 2m$. Thus $p^{3n} \leq p^m$. This contradicts $p^m \leq q^2$. It follows that $p^m \leq q$, i.e., $m \leq n$. According to Lemmas 4 and 5, we can take $t \leq 4$. Thus $6n \leq mt \leq 4m \leq 4n$. With this contradiction, the proof of Theorem 3 is complete. □

In conclusion, we note that to show that our proof of Theorem 3 remains valid if $PSU(3, q^2)$ is replaced by $Sz(q)$ or ${}^2G_2(q)$, it is necessary only to prove an appropriate version of Lemmas 4 and 5. I leave the case ${}^2G_2(q)$ as a problem. For the Suzuki groups, the following result, a corollary of [**10**, Theorem 9], is easily seen to suffice.

LEMMA 6. *Let P_1 be a nontrivial elementary abelian 2-subgroup of $Sz(q)$. Then there exists a subgroup P_2 conjugate to P_1 such that $\langle P_1, P_2 \rangle$ contains a dihedral group of order $2(q - 1)$ if $q > 2$, of order 10 if $q = 2$, which is maximal in $Sz(q)$ (so that $\langle P_1, P_2 \rangle = Sz(q)$ if $|P_1| > 2$).*

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