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A VOLTERRA EQUATION WITH SQUARE INTEGRABLE SOLUTION

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ABSTRACT. We study the asymptotic behavior of the solutions of the nonlinear Volterra integrodifferential equation

$$x'(t) + \int_0^t a(t-s)g(x(s))\,ds = f(t) \qquad (t \in R^+).$$

Here $R^+ = [0, \infty)$, a, g and f are given real functions, and x is the unknown solution. In particular, we give sufficient conditions which imply that x and x' are square integrable.

1. Introduction and summary of results. We study the asymptotic behavior of the solutions of the Volterra integrodifferential equation

$$x'(t) + \int_0^t a(t-s)g(x(s))\,ds = f(t) \qquad (t \in R^+). \tag{1.1}$$

Here $R^+ = [0, \infty)$, the prime denotes differentiation, a, g and f are given, real functions, and x is the unknown solution. In particular, we give sufficient conditions which imply that the solutions satisfy $x, x' \in L^2(R^+)$.

Our assumptions are the following:

$$a = b + c \textit{ is strongly positive definite}, \tag{1.2}$$

where

$$\begin{aligned}&b \in L^1(R^+) \textit{ satisfies } |\hat{b}(\omega)|^2 \leqslant \beta \operatorname{Re} \hat{b}(\omega) \quad (\omega \in R)\\ &\textit{for some } \beta \geqslant 0,\end{aligned} \tag{1.3}$$

$$c \textit{ is positive definite, and } c' \in L^1 \cap \mathrm{BV}(R^+), \tag{1.4}$$

$$g \in C(R),\ \xi g(\xi) > 0 \quad (\xi \neq 0), \textit{ and } \liminf_{\xi \to 0} g(\xi)/\xi > 0, \tag{1.5}$$

$$\begin{aligned}&f = f_1 + f_2 + f_3, \textit{ where } f_1 \in L^2(R^+), f_2 \in \mathrm{BV}(R^+),\\ &\textit{and } f_3 \in L^\infty(R^+), f_3' \in L^2(R^+).\end{aligned} \tag{1.6}$$

Here $\hat{b}(\omega) = \int_0^\infty e^{-i\omega t}\, b(t)\,dt$ is the Fourier transform of b. The strong positive definiteness of a means that there exists $\epsilon > 0$ such that the function $a(t) - \epsilon e^{-t}$ is positive definite. The statements concerning c' and f_3' should be interpreted as requirements that c, f_3 be locally absolutely continuous, together with the additional conditions on the derivatives. BV stands for functions of bounded variation.

We call x a solution of (1.1) if x is locally absolutely continuous, and (1.1) holds a.e. In addition to (1.2)–(1.6) above we shall have to assume that a solution x of

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(1.1) satisfies

$$x, Q_a \in L^\infty(R^+), \tag{1.7}$$

where

$$Q_a(T) = \int_0^T g(x(t)) \int_0^t a(t-s) g(x(s))\, ds\, dt \qquad (T \in R^+). \tag{1.8}$$

Sufficient conditions for (1.7) to hold can be found in [7]. For example, any one of (1.9)–(1.11) below combined with (1.2)–(1.5) and the assumption

$$-\int_{-\infty}^0 g(\xi)\, d\xi = \int_0^\infty g(\xi)\, d\xi = \infty$$

imply (1.7):

$$f \in L^1(R^+), \textit{ and } \limsup_{|\xi|\to\infty} |g(\xi)| \left(1 + \int_0^\xi g(\eta)\, d\eta\right)^{-1} < \infty, \tag{1.9}$$

$$f, f' \in L^2(R^+), \tag{1.10}$$

$$c(\infty) > 0, \textit{ and } f \in \mathrm{BV}(R^+). \tag{1.11}$$

We prove the following result:

THEOREM 1. *Let* (1.2)–(1.6) *hold, and let x be a solution of* (1.1) *satisfying* (1.7). *Then $x, x' \in L^2(R^+)$.*

Theorem 1 is an improved version of [**8**, Theorem 1(iii)]. One gets [**8**, Theorem 1(iii)] by adding (1.10) and

$$b \equiv 0, \qquad c - c(\infty) \in L^1(R^+) \tag{1.12}$$

to the hypothesis of Theorem 1.

Theorem 1 extends some of the results in [5] and [6]. The hypothesis used here is comparatively strong, but, on the other hand, we now get the stronger conclusion $x \in L^2(R^+)$ (which amounts to a faster convergence of x to zero than [5] and [6] yield).

This work may be regarded as a strengthening of [8], which in turn was inspired by some estimates in the two papers [1] and [2] of MacCamy. In spite of this fact our argument is quite different from MacCamy's. MacCamy does not work with a scalar equation as we do, but with an abstract Volterra equation of hyperbolic type. We shall return elsewhere [10] to the question of how the estimates in the proof of Theorem 1 should be modified in the abstract case.

We discuss conditions (1.2)–(1.4) in §3.

2. Proof of Theorem 1. Define

$$\varphi(t) = g(x(t)), \qquad d(t) = (1 + c(0))e^{-t} \qquad (t \in R^+).$$

Let $*$ denote convolution, subtract $d * \varphi$ from both sides of (1.1), and use (1.2), (1.6) to get

$$x' - (d-c) * \varphi - f_2 - f_3 = f_1 - (b+d) * \varphi. \tag{2.1}$$

Define

$$u = (d-c) * \varphi, \qquad v = (d-c)' * \varphi, \qquad w = (b+d) * \varphi. \tag{2.2}$$

Multiply (2.1) by x', integrate over $(0, T)$, and integrate the terms on the left-hand side by parts (except the first one) to get

$$\begin{aligned}\int_0^T [x'(t)]^2\,dt + \int_0^T x(t)g(x(t))\,dt &= x(T)[u(T) + f_2(T) + f_3(T)] - x(0)f_3(0)\\ &\quad - \int_0^T x(t)\,df_2(t) + \int_0^T x'(t)[f_1(t) - w(t)]\,dt\\ &\quad - \int_0^T x(t)[v(t) + f_3'(t)]\,dt,\end{aligned}$$

where we have redefined f_2 so that it is continuous from the left, and $f_2(0) = 0$. By the Hölder and Minkowski inequalities,

$$\begin{aligned}\|x'\|_2^2 + \int_0^T x(t)g(x(t))\,dt &\leqslant (\|u\|_\infty + \|f_2\|_\infty + |||f_2||| + 2\|f_3\|_\infty)\|x\|_\infty\\ &\quad + (\|f_1\|_2 + \|w\|_2)\|x'\|_2 + (\|v\|_2 + \|f_3'\|_2)\|x\|_2,\end{aligned} \tag{2.3}$$

where $\|\ \|_p$ $(p = 2, \infty)$ is the norm of $L^p(0, T)$, and $|||f_2|||$ is the total variation of f_2.

We claim that

$$u \in L^\infty(R^+), \qquad v, w \in L^2(R^+). \tag{2.4}$$

Assume this for the moment. Then, by (1.6), (1.7) and (2.4), inequality (2.3) is of the form

$$\|x'\|_2^2 + \int_0^T x(t)g(x(t))\,dt \leqslant C(1 + \|x'\|_2 + \|x\|_2), \tag{2.5}$$

where C is a (sufficiently large) constant independent of T. Observe that (1.5), (1.7) imply the existence of $\epsilon > 0$ such that $x(t)g(x(t)) \geqslant \epsilon|x(t)|^2$ $(t \in R^+)$. Hence (2.5) becomes

$$\|x'\|_2^2 + \epsilon\|x\|_2^2 \leqslant C(1 + \|x'\|_2 + \|x\|_2),$$

from which the conclusion of Theorem 1 follows.

It remains to verify the crucial estimate (2.4). Observe that the functions b, c and d are all positive definite, and that by (1.2), $0 \leqslant Q_b(T) \leqslant Q_a(T)$, $0 \leqslant Q_c(T) \leqslant Q_a(T)$, and $0 \leqslant Q_d(t) \leqslant CQ_a(T)$, where Q_b, Q_c and Q_d are defined as in (1.8), and C is some positive constant. Thus (1.7) implies

$$Q_b, Q_c, Q_d \in L^\infty(R^+). \tag{2.6}$$

Both c and d are continuous and positive definite, and so [4, Lemma 6.1] yields

$$|c * \varphi(T)|^2 \leqslant 2c(0)Q_c(T), \qquad |d * \varphi(T)|^2 \leqslant 2d(0)Q_d(T). \tag{2.7}$$

Combining (2.2) with (2.6) and (2.7) one gets the first part of (2.4). By (1.3) and [5, Lemma 1],

$$\|b * \varphi\|_2^2 \leqslant \beta Q_b(T). \tag{2.8}$$

Observe that $c', d, d' \in L^1 \cap \mathrm{BV}(R^+)$, and use (1.2) and [9, Lemma 2.2] to get

$$\|c' * \varphi\|_2^2 + \|d * \varphi\|_2^2 + \|d' * \varphi\|_2^2 \leqslant CQ_a(T),$$

for some constant C. Combining this with (1.7), (2.2), (2.6) and (2.8) we get the second part of (2.4). This completes the proof of Theorem 1.

3. Comments. The proof of Theorem 1 gives us, in fact, a little more than $x \in L^2(R^+)$, namely

$$\int_0^\infty x(t)g(x(t))\,dt < \infty \tag{3.1}$$

(cf. (2.5)). If $\limsup_{\xi\to 0} g(\xi)/\xi < \infty$, then (3.1) is equivalent to $x \in L^2(R^+)$. However, if e.g., $g(\xi) = \xi^{1/3}$ (which satisfies (1.5)), then (3.1) becomes $x \in L^{4/3}(R^+)$.

The conditions (1.2)–(1.4) require a splitting of a into two parts, and given a it is not always obvious how this splitting should be done. Some requirements are obvious: If a or a' is unbounded, then the unbounded part must go into b, and if a is not integrable, then the nonintegrable part must go into c. Below we shall give some examples where the splitting succeeds. For example, in the following two cases (1.3) holds:

$$b \in L^1 \cap \mathrm{BV}(R^+) \textit{ is strongly positive definite}, \tag{3.2}$$

$$b \in L^1(R^+), \textit{ and } b,\ -b' \textit{ are convex} \tag{3.3}$$

(see [**5**, Theorem **2**]). Thus, if, e.g., a is strongly positive definite, and $a - a(\infty) \in L^1 \cap \mathrm{BV}(R^+)$, then one can take $b = a - a(\infty)$, $c = a(\infty)$ (the strong positive definiteness of a implies the strong positive definiteness of b in this case, and $a(\infty) \geqslant 0$). On the other hand, if $a' \in L^1 \cap \mathrm{BV}(R^+)$, then one may choose $b \equiv 0$, $c = a$. An example where (3.3) is used is the following: Suppose that $a(t) = t^{-\alpha}$ ($t \in R^+$), where $0 < \alpha < 1$, and define $b(t) = t^{-\alpha} - (1+t)^{-\alpha}$, $c(t) = (1+t)^{-\alpha}$ (cf. [**3**, Corollary 2.2]).

One way of simplifying the problem of how one should split a into $b + c$ is to modify (1.3), (1.4), and modify the proof of Theorem 1 accordingly. One can replace (1.3), (1.4) by

$$\begin{array}{l} b \in L^1(R^+), \textit{ and } |\hat{b}(\omega)|^2 \leqslant \beta \operatorname{Re} \hat{a}(\omega)\ (\omega \in R) \\ \textit{for some } \beta > 0, \end{array} \tag{3.4}$$

$$c \in L^2(R^+), \qquad c' \in L^1 \cap \mathrm{BV}(R^+). \tag{3.5}$$

Most of the proof of Theorem 1 remains valid. (2.6) should be replaced by

$$Q_e \in L^\infty(R^+),$$

where $e(t) = e^{-t}$ ($t \in R^+$), and (2.7), (2.8) by

$$|c * \varphi(T)|^2 \leqslant 2Q_e(T)\int_0^\infty \left(c^2(t) + [c'(t)]^2\right) dt,$$

$$|d * \varphi(T)|^2 \leqslant 2(1 + c(0))^2 Q_e(T), \tag{3.6}$$

$$\|b * \varphi\|_2^2 \leqslant \beta Q_a(T). \tag{3.7}$$

The proofs of (3.6), (3.7) are similar to the proofs of [**4**, Lemma 6.1] and [**5**, Lemma 1].

In (3.4), (3.5) it is no longer required that b and c be positive definite, which clearly facilitates the splitting of a into $b + c$. In particular, (3.2) can be weakened to $b \in L^1 \cap \mathrm{BV}(R^+)$. On the other hand, the added condition $c \in L^2(R^+)$ prevents the use of (3.4), (3.5), e.g., when $a(t) = t^{-\alpha}$ with $0 < \alpha < \frac{1}{2}$. Also observe that (3.4), (3.5) exclude the possibility $a(\infty) > 0$.

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