HOW TO RECOGNIZE A LOCALIZED SPHERE

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ABSTRACT. Two cohomological characterizations of the sphere localized at a prime
are given.

The purpose of this note is to obtain necessary and sufficient conditions on the cohomology of a nilpotent space in order to insure that it has the homotopy type of S_p^m, the sphere localized at the prime p (cf. [2]). p will denote a fixed prime integer. $\hat{\mathbf{Z}}_p$, $\mathbf{Z}_{(p)}$, $\mathbf{Z}_n$, $\mathbf{Z}_{p^\infty}$ denote the p-adic integers, the integers localized at p, the integers modulo n and the p^mth roots of unity with m running over all integers > 0, respectively. Group will always mean abelian group. A group is called p-local if it is q-divisible for every $q \neq p$ and has no q-torsion for every $q \neq p$. A nilpotent space is called p-local if its homology groups with integral coefficients are p-local [2, p. 53].

Recall that a subgroup B of a group A is called p-basic if the following holds:

(i) B is a direct sum of cyclic p-groups and infinite cyclic groups,

(ii) A/B is p-divisible,

(iii) B/p^kB is a direct summand of A/p^kB for all k.

Every group contains p-basic subgroups [1, I, p. 137].

We use freely the most elementary properties of the functors Hom and Ext. A standard reference is [1, Chapters VIII and IX]. In particular we shall use the following:

$$\text{Ext}(\mathbf{Z}_n, A) = A/nA, \quad \text{Ext}(\mathbf{Z}_{p^\infty}, \mathbf{Z}_p) = \mathbf{Z}_p, \quad \text{Ext}(\mathbf{Q}, \mathbf{Z}_{(p)}) = \mathbf{Q}^{\aleph_0}, \quad \text{Ext}(\mathbf{Z}_{p^\infty}, \mathbf{Z}_{(p)}) = \hat{\mathbf{Z}}_p.$$

All spaces are assumed to be of the homotopy type of CW-complexes.

LEMMA 1. *Let B be a p-basic subgroup of A. Then $\text{Hom}(A, \mathbf{Z}_p) = \text{Hom}(B, \mathbf{Z}_p)$.*

PROOF. Since A/B is p-divisible and $\mathbf{Z}_p$ contains no p-divisible subgroups other than 0, the Hom-Ext exact sequence associated to $B \rightarrowtail A \twoheadrightarrow A/B$ yields that $\text{Hom}(A, \mathbf{Z}_p)$ is a subgroup of $\text{Hom}(B, \mathbf{Z}_p)$. Let $f\colon B \to \mathbf{Z}_p$. Clearly, f factorizes through $f'\colon B/pB \to \mathbf{Z}_p$. Since B is a p-basic subgroup of A, B/pB is a direct summand of A/pB. Hence f' extends to A/pB and yields a homomorphism $g \in \text{Hom}(A, \mathbf{Z}_p)$. $\square$

LEMMA 2. *If A is a p-local group such that $\text{Hom}(A, \mathbf{Q}) = \text{Hom}(A, \mathbf{Z}_p) = 0$, then A is a divisible p-group.*

PROOF. Since $\text{Hom}(A, \mathbf{Q}) = 0$, A has no elements of infinite order. Since A is p-local, A is a p-group. Let B be a p-basic subgroup. By Lemma 1, $\text{Hom}(B, \mathbf{Z}_p) = \text{Hom}(A, \mathbf{Z}_p) = 0$ and since B is a direct sum of cyclic p-groups then $B = 0$. That is A is p-divisible and so, since A is p-local, A is divisible. $\square$

LEMMA 3. *Let E be a nilpotent space such that $H_*(E; \mathbf{Z}) = H_*(S_p^m; \mathbf{Z})$. Then E has the homotopy type of S_p^m.*

PROOF. If $m > 1$ then $H_1 E = 0$. In other words, the abelianization of $\pi_1 E$ is trivial. Since $\pi_1 E$ is a nilpotent group we conclude that E is simply connected and so E is a Moore space $M(\mathbf{Z}_{(p)}, m)$, that is a localized sphere S_p^m.
If $m = 1$ then we have

$$\left[E, S_p^1 \right] = \left[E, K(\mathbf{Z}_{(p)}, 1) \right] \cong H^1(E; \mathbf{Z}_{(p)}) = \mathbf{Z}_{(p)}$$

where $K(\mathbf{Z}_{(p)}, 1)$ denotes an Eilenberg-Mac Lane space of type $(\mathbf{Z}_{(p)}, 1)$. Then there exists a map $f: E \to S_p^1$ which induces isomorphisms in homology. Because E and S_p^1 are nilpotent spaces, f must be a homotopy equivalence. The result follows. $\square$

THEOREM 4. *Let E be a nilpotent space such that*
(1) *E is p-local,*
(2) *$H^*(E; R) = H^*(S^m; R)$ where $R = \mathbf{Z}_p$ or $R = \hat{\mathbf{Z}}_p$,*
(3) *$H^*(E; \mathbf{Q}) = H^*(S^m; \mathbf{Q})$,*
(4) *$H^{m+1}(E; \mathbf{Z}_{(p)}) = 0$.*
Then $E = S_p^m$. If any of the above conditions is omitted then the conclusion is false.

PROOF. It suffices to prove that $H_*(E; \mathbf{Z}) = H_*(S_p^m; \mathbf{Z})$. Let us consider the exact sequence $\hat{\mathbf{Z}}_p \overset{\cdot p}{\rightarrowtail} \hat{\mathbf{Z}}_p \twoheadrightarrow \mathbf{Z}_p$. It induces a long exact sequence in cohomology:

$$\ldots \to H^r\!\left(E; \hat{\mathbf{Z}}_p\right) \overset{\cdot p}{\to} H^r\!\left(E; \hat{\mathbf{Z}}_p\right) \to H^r\!\left(E; \mathbf{Z}_p\right) \to H^{r+1}\!\left(E; \hat{\mathbf{Z}}_p\right) \to \ldots.$$

From this it follows easily that we may assume, without loss of generality that $R = \mathbf{Z}_p$ in condition (2). On the other hand, condition (3) implies that $H_r E$ has torsion-free rank zero if $r \neq m$ and has torsion-free rank one if $r = m$.
The universal coefficient theorem:

$$\text{Ext}\!\left(H_{r-1}E, \mathbf{Z}_p\right) \rightarrowtail H^r(E; \mathbf{Z}_p) \twoheadrightarrow \text{Hom}(H_r E, \mathbf{Z}_p)$$

yields: $\text{Hom}(H_r E, \mathbf{Z}_p) = 0$ if $r \neq m$ and $\text{Ext}(H_r E, \mathbf{Z}_p) = 0$ if $r \neq m - 1$. It follows from Lemma 2 that $H_r E$ is a divisible p-group if $r \neq m$. The structure theorem for divisible groups [1, I, p. 104] shows that $H_r E = \oplus \mathbf{Z}_{p^\infty}$. But $\text{Ext}(H_r E, \mathbf{Z}_p) = 0$ if $r \neq m - 1$, hence $H_r E = 0$ if $r \neq m, m - 1$ because $\text{Ext}(\mathbf{Z}_{p^\infty}, \mathbf{Z}_p) = \mathbf{Z}_p$.
Since $H^m(E; \mathbf{Z}_p) = \mathbf{Z}_p$ the universal coefficient theorem shows that there are two cases:
(A) $\text{Ext}(H_{m-1}E, \mathbf{Z}_p) = \mathbf{Z}_p$, $\text{Hom}(H_m E, \mathbf{Z}_p) = 0$.
$H_{m-1}E$ is a divisible p-group and so $H_{m-1}E = \oplus \mathbf{Z}_{p^\infty}$. Since $\text{Ext}(H_{m-1}E, \mathbf{Z}_p) = \mathbf{Z}_p$ it follows that $H_{m-1}E = \mathbf{Z}_{p^\infty}$. On the other hand, let B be a p-basic subgroup of $H_m E$. By Lemma 1, $\text{Hom}(B, \mathbf{Z}_p) = \text{Hom}(H_m E, \mathbf{Z}_p) = 0$ and so $B = 0$ and $H_m E$ is divisible because it is p-local and p-divisible. Since $H_m E$ has torsion-free rank one and $\text{Ext}(H_m E, \mathbf{Z}_p) = 0$ we conclude that $H_m E = \mathbf{Q}$. Hence, we see that in this case

the space E has the homology of $M(\mathbf{Q}, m) \vee M(\mathbf{Z}_{p^\infty}, m - 1)$, the wedge of the Moore spaces $M(\mathbf{Q}, m)$ and $M(\mathbf{Z}_{p^\infty}, m - 1)$.

(B) $\mathrm{Ext}(H_{m-1}E, \mathbf{Z}_p) = 0$, $\mathrm{Hom}(H_m E, \mathbf{Z}_p) = \mathbf{Z}_p$.

$H_{m-1}E$ is a divisible p-group such that $\mathrm{Ext}(H_{m-1}E, \mathbf{Z}_p) = 0$. Since $\mathrm{Ext}(\mathbf{Z}_{p^\infty}, \mathbf{Z}_p)$ $= \mathbf{Z}_p$ then $H_{m-1}E = 0$ and so E has homology $\neq 0$ only in dimension m. Let B be a p-basic subgroup of $H_m E$. We have $\mathrm{Hom}(B, \mathbf{Z}_p) = \mathrm{Hom}(H_m E, \mathbf{Z}_p) = \mathbf{Z}_p$. Since B is a direct sum of infinite cyclic groups and cyclic p-groups, either $B = \mathbf{Z}$ or $B = \mathbf{Z}_{p^k}$. If $B = \mathbf{Z}_{p^k}$ then condition (iii) in the definition of p-basic subgroup yields that $\mathbf{Z}_{p^k}$ is a direct summand of $H_m E$, but this is impossible because $\mathrm{Ext}(H_m E, \mathbf{Z}_p)$ $= 0$. Hence $\mathbf{Z}$ is a p-basic subgroup of $H_m E$. Since $D = H_m E / \mathbf{Z}$ is divisible and $\mathrm{Ext}(H_m E, \mathbf{Z}_p) = 0$, the Hom-Ext exact sequence associated to $\mathbf{Z} \rightarrowtail H_m E \twoheadrightarrow D$ gives $\mathrm{Ext}(D, \mathbf{Z}_p) = 0$. If we apply now the structure theorem for divisible groups we obtain that D has no p-torsion and so $H_m E$ is torsion-free because it is p-local and so it can only have p-torsion. Then $H_m E$ is a p-local torsion-free group of rank one. We apply now the classification theorem for these groups [1, II, p. 110]. Since $H_m E$ is p-local, in order to prove that $H_m E = \mathbf{Z}_{(p)}$ it suffices to show that $H_m E$ contains elements of p-height zero. Let us consider $1 \in \mathbf{Z} \subset H_m E$. If $1 = pa$, $a \in H_m E$ then in D we have $0 = p\bar{a}$ and since D has no p-torsion we get $a \in \mathbf{Z}$, a contradiction. Hence, the type of $H_m E$ is $t_q = \infty$ if $q \neq p$ and $t_p = 0$. Then the space E has the same integral homology as S_p^m and then, by Lemma 3, $E = S_p^m$.

We have proved that if a nilpotent space E satisfies conditions (1), (2), (3) of the theorem then either $E = S_p^m$ or E has the same homology as $M(\mathbf{Q}, m) \vee M(\mathbf{Z}_{p^\infty}, m - 1)$. It suffices to show that if $H_m E = \mathbf{Q}$ then E does not satisfy condition (4). This follows easily from the fact $\mathrm{Ext}(\mathbf{Q}, \mathbf{Z}_{(p)}) = \mathbf{Q}^{\aleph_0}$.

Finally, note that none of the conditions (1), (2), (3), (4) can be omitted. Let us consider the spaces $E_1 = S_p^m \vee M(\mathbf{Z}_q, 2)$ (q a prime $\neq p$); $E_2 = S_p^m \vee M(\mathbf{Z}_p, 2m)$; $E_3 = M(\mathbf{Z}_{p^\infty}, m - 1)$; $E_4 = M(\mathbf{Q}, m) \vee M(\mathbf{Z}_{p^\infty}, m - 1)$. It is easily seen that the space E_i verifies conditions (1), (2), (3), (4) except the ith, but $E_i \neq S_p^m$. $\square$

The above theorem shows that the cohomology with coefficients $\mathbf{Z}_p$, $\mathbf{Z}_p$, $\mathbf{Q}$ is not enough to characterize the localized spheres. The following theorem shows that the cohomology with coefficients $\mathbf{Z}_{(p)}$ is suitable for this purpose.

THEOREM 5. *Let E be a p-local nilpotent space such that $H^*(E; \mathbf{Z}_{(p)}) = H^*(S^m; \mathbf{Z}_{(p)})$. Then $E = S_p^m$.*

PROOF. From the universal coefficient theorem we obtain $\mathrm{Hom}(H_r E, \mathbf{Z}_{(p)}) = 0$ if $r \neq m$ and $\mathrm{Ext}(H_r E, \mathbf{Z}_{(p)}) = 0$ if $r \neq m - 1$. Further, $\mathrm{Ext}(H_{m-1}E, \mathbf{Z}_{(p)})$ is a p-local subgroup of $\mathbf{Z}_{(p)}$. Then either $\mathrm{Ext}(H_{m-1}E, \mathbf{Z}_{(p)}) = \mathbf{Z}_{(p)}$ or $\mathrm{Ext}(H_{m-1}E, \mathbf{Z}_{(p)}) = 0$. But in the first case $\mathbf{Z}_{(p)}$ will be a cotorsion group (cf. [1, I, p. 232] and Theorem 54.6 of [1, I, p. 235]) and this is impossible because $\mathrm{Ext}(\mathbf{Q}, \mathbf{Z}_{(p)}) = \mathbf{Q}^{\aleph_0} \neq 0$. Hence, $\mathrm{Ext}(H_r E, \mathbf{Z}_{(p)}) = 0$ for all r.

Let B_r be a p-basic subgroup of $H_r E$. If B_r contains a direct summand of the form $\mathbf{Z}_{p^k}$ then it follows easily from the definition of p-basic subgroup that $\mathbf{Z}_{p^k}$ is a direct summand of $H_r E$. But $\mathrm{Ext}(\mathbf{Z}_{p^k}, \mathbf{Z}_{(p)}) = \mathbf{Z}_{(p)}/p^k \mathbf{Z}_{(p)} \neq 0$. Hence, B_r is free. Let us consider the Hom-Ext exact sequence associated to $\mathbf{Z}_{(p)} \overset{p}{\rightarrowtail} \mathbf{Z}_{(p)} \twoheadrightarrow \mathbf{Z}_p$:

$$0 \to \mathrm{Hom}(H_rE, \mathbf{Z}_{(p)}) \xrightarrow{\cdot p} \mathrm{Hom}(H_rE, \mathbf{Z}_{(p)}) \to \mathrm{Hom}(H_rE, \mathbf{Z}_p) \to 0.$$

We get that $\mathrm{Hom}(H_rE, \mathbf{Z}_p) = 0$ if $r \neq m$. By Lemma 1, $\mathrm{Hom}(B_r, \mathbf{Z}_p) = \mathrm{Hom}(H_rE, \mathbf{Z}_p) = 0$, hence $B_r = 0$ if $r \neq m$. This shows that H_rE is a divisible group for $r \neq m$. But the structure theorem for divisible groups and the equalities $\mathrm{Ext}(\mathbf{Q}, \mathbf{Z}_{(p)}) = \mathbf{Q}^{\aleph_0}$, $\mathrm{Ext}(\mathbf{Z}_{p^\infty}, \mathbf{Z}_{(p)}) = \hat{\mathbf{Z}}_p$, $\mathrm{Ext}(H_rE, \mathbf{Z}_{(p)}) = 0$ yield $H_rE = 0$ if $r \neq m$.

In the case $r = m$ the above exact sequence shows that $\mathrm{Hom}(H_mE, \mathbf{Z}_p) = \mathbf{Z}_p$. By Lemma 1, $B_m = \mathbf{Z}$. The Hom-Ext exact sequence associated to $\mathbf{Z} \rightarrowtail H_mE \twoheadrightarrow H_mE/\mathbf{Z}$ shows that $\mathrm{Ext}(H_mE/\mathbf{Z}, \mathbf{Z}_{(p)})$ is the image of the countable group $\mathrm{Hom}(\mathbf{Z}, \mathbf{Z}_{(p)})$. Since $\mathrm{Ext}(\mathbf{Q}, \mathbf{Z}_{(p)})$ and $\mathrm{Ext}(\mathbf{Z}_{p^\infty}, \mathbf{Z}_{(p)})$ are both uncountable and since $H_mE/\mathbf{Z}$ is a divisible group, we get that $H_mE/\mathbf{Z}$ is a torsion group without p-torsion. This leads to $\mathrm{Hom}(H_mE/\mathbf{Z}, \mathbf{Q}) = 0$ and so $\mathrm{Hom}(H_mE, \mathbf{Q}) = \mathrm{Hom}(\mathbf{Z}, \mathbf{Q}) = \mathbf{Q}$. Hence H_mE is a group of torsion-free rank one. Further, H_mE is torsion-free because $H_mE/\mathbf{Z}$ has no p-torsion and H_mE is p-local. Now we can show, as in the proof of Theorem 4 that $H_mE = \mathbf{Z}_{(p)}$ and so the space E has the same integral homology as S_p^m. Then, by Lemma 3, E is a localized sphere S_p^m. $\qquad\square$

References

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