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MAXIMAL SYMMETRY AND FULLY WOUND COVERINGS

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ABSTRACT. A compact bordered Klein surface of genus $g \geqslant 2$ is said to have *maximal symmetry* if its automorphism group is of order $12(g-1)$, the largest possible. We show that for each value of k there are only finitely many topological types of bordered Klein surfaces with maximal symmetry that have exactly k boundary components. We also prove that there are no bordered Klein surfaces with maximal symmetry that have exactly p boundary components for any prime $p \geqslant 5$. These results are established using the concept of a *fully wound covering*, that is, a full covering $\varphi: X \to Y$ of the bordered surface Y with the maximum possible boundary degree.

0. Introduction. A compact bordered Klein surface of genus $g \geqslant 2$ has at most $12(g-1)$ automorphisms [5]. A bordered surface for which the bound $12(g-1)$ is attained is said to have *maximal symmetry*. Numerous examples of surfaces with maximal symmetry are in [3], [5], and [6]. In fact, given a single surface with maximal symmetry, there are techniques for constructing infinite families of surfaces with maximal symmetry [3]. However, there are infinitely many values of the genus g for which there is no surface with maximal symmetry [7].

In this paper we are concerned with the number of boundary components of a surface with maximal symmetry. For each value of the positive integer k there are, of course, infinitely many topological types of Klein surfaces with k boundary components. We show that only finitely many of these have maximal symmetry. To do so we bound the size of the automorphism group of a surface with maximal symmetry in terms of the number of its boundary components. We use the relationship between the topology of the surface and the structure of its automorphism group as well as the concept of a *fully wound covering*, that is, a full covering $\varphi: X \to Y$ of the bordered surface Y such that each component of ∂X is wound around its image the largest possible number of times. Along the way we establish some facts about fully wound coverings of surfaces with maximal symmetry.

Another natural problem is the determination of the possible values for the number of boundary components of a surface with maximal symmetry. We do not have a complete answer, but we show that there are no surfaces with maximal symmetry with p boundary components for any prime $p \geqslant 5$. Here too the concept of a fully wound covering is useful.

1. M^*-groups and regular maps. For any Klein surface X, let $A(X)$ denote the group of automorphisms of X. We assume that all surfaces are compact.

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A finite group G is called an *M^*-group* [6] if it is generated by three distinct elements t, u, v of order two which satisfy the relations

$$(tu)^2 = (tv)^3 = 1. \tag{1.1}$$

The main result of [6] is that G is an M^*-group if and only if G is the automorphism group of a bordered Klein surface with maximal symmetry.

The order of uv is called an *index* of G. An M^*-group may have more than one index, corresponding to different sets of generators; for an example, see [3, §2]. We shall need the following, which gives one method for finding a different set of generators.

LEMMA 1. *Let G be an M^*-group with index q and generators t, u, v satisfying the relations* (1.1). *If $o(tuv) = r$, then G also has index r.*

PROOF. Let $u' = tu$. Then t, u', v generate G, satisfy the necessary relations, and $o(u'v) = r$. Note that $o(tu'v) = q$.

There is an important correspondence between bordered Klein surfaces with maximal symmetry and regular maps [3]. For the basic definitions on regular maps, see [3, §6] and [1, pp. 20, 101–103]. We use *regular* in the strong sense of [3]. A map is said to be of type $\{p, q\}$ if it is composed of p-gons, q meeting at each vertex. Now let X be a bordered Klein surface with maximal symmetry of genus $g > 2$. Then X corresponds to a regular map M of type $\{3, q\}$ on the surface X^* obtained from X by attaching a disc to each boundary component. If X has k boundary components, then $q = 6(g - 1)/k$. The automorphism group G of X is isomorphic to the automorphism group of the regular map M, and moreover G has index q. In particular note that $o(G) = 12(g - 1) = 2qk$. For the details, see [3, §6].

Let G have generators t, u, v that satisfy the relations (1.1) with $o(uv) = q$. Write $x = tu$, $z = uv$, so that $t = ux$, $v = uz$. Then G is generated by u, x, z which satisfy the relations

$$u^2 = x^2 = z^q = (ux)^2 = (uz)^2 = (xz)^3 = 1. \tag{1.2}$$

We can use the map M on X^* to represent the group G as follows. The figure (drawn with $q = 6$) shows part of the map on X^*, with the boundary components of X centered about the vertices of M. Choose a vertex a of the map. Let r be the automorphism that cyclically permutes the edges of a face containing a, as shown. Then the automorphism z cyclically permutes the edges meeting at the vertex a, u is a reflection about the edge ab, and $x = rz$ is the half-turn which fixes the edge ab, but interchanges a and b (see [1, pp. 101, 102] where R, S, R_3 correspond to our r, z, u).

Let C be the boundary component of X that is centered about the vertex a. Now think of G as acting on X, and let $S = \{f \in G | f(C) = C\}$ be the subgroup of G that fixes C. Since G acts transitively on the components of ∂X, $[G: S] = k$ and $o(S) = 2q$. Then clearly $S = \langle u, z\rangle$, and S is isomorphic to the dihedral group of order $2q$. We shall often use this characterization of the subgroup $\langle u, z\rangle$.

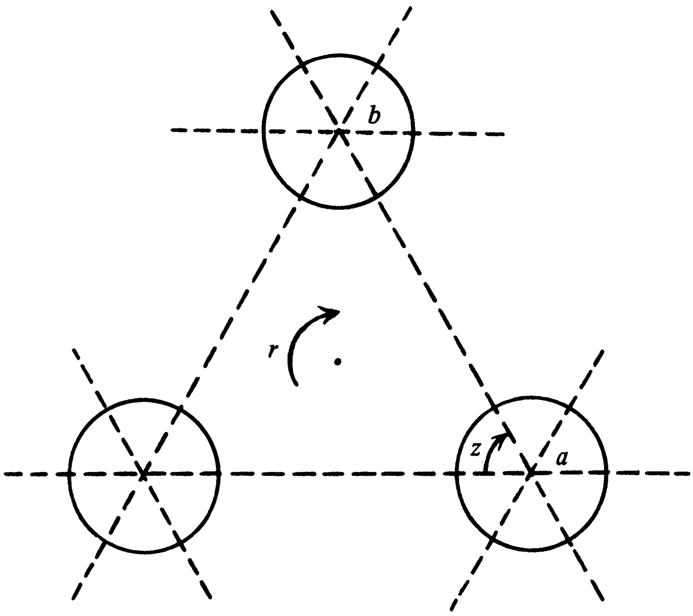

FIGURE 1

2. Maximal symmetry and coverings. Let φ: $X \to X'$ be a nonconstant morphism of bordered Klein surfaces. We call φ a *smooth covering* if it is unramified and without folding and a *normal covering* if the covering transformations act transitively on fibers. If φ is smooth and normal and every automorphism of X' lifts to an automorphism of X, then φ is called a *full covering*. The basic results about full coverings of surfaces with maximal symmetry are the following.

THEOREM A [3]. *Let φ: $X \to X'$ be a full covering of the bordered Klein surface X'. If X' has maximal symmetry, then so does X. In this case, $A(X') = A(X)/N$, where N is the group of covering transformations.*

THEOREM B [3]. *Let X be a bordered Klein surface with maximal symmetry and let N be a normal subgroup of $G = A(X)$ of index larger than 6. Let $X' = X/N$, $G' = G/N$, and let φ: $X \to X'$ be the quotient map. Then X' has maximal symmetry with automorphism group G', and φ is a full covering.*

An M^*-group is said to be *M^*-simple* [3] if it has no proper M^*-quotient groups. If X has maximal symmetry and $A(X)$ is M^*-simple, then we say that X has *primitive maximal symmetry*. It follows from Theorem B that if X has maximal symmetry, then it is a full covering of a surface X' with primitive maximal symmetry. Thus the surfaces with primitive maximal symmetry are of special interest (see [3, §3]).

Let φ: $X \to X'$ be a smooth and normal covering of degree r, and let C_i be the ith component of ∂X. The *local boundary degree* d_i is the number of times C_i wraps around its image in $\partial X'$. If all of the d_i are equal, then the common value is called the (*global*) *boundary degree* of φ. In general, a covering does not have a global boundary degree, but if φ is a full covering and the surface X' has maximal symmetry, then the boundary degree is well-defined [3, §3].

Now let the covering φ have boundary degree d. If k and k' are the number of components of ∂X and $\partial X'$ respectively, then

$$kd = k'r, \tag{2.1}$$

and clearly the boundary degree d divides r. In case $d = r$, we say that the covering φ has *maximal winding*, since each component of ∂X is wound around its image r times, the maximum possible. A full covering with maximal winding will be called a *fully wound covering*. For several examples of fully wound coverings see the table in [3, §4]. This type of covering will be quite useful in the following sections.

Suppose the covering φ: $X \to X'$ is fully wound, and let N be the group of covering transformations, so that $X' = X/N$. Then clearly N fixes each component of ∂X, and N is the cyclic group of order r. Of course $k = k'$.

3. Fully wound coverings. Let X be a Klein surface with k boundary components that has maximal symmetry. Let $G = A(X)$ have index q and generators u, x, z satisfying the relations (1.2). We make free use of these relations throughout this section. The following lemma and part of the proof of Theorem 2 were suggested by a result of Newman [8, p. 270] on the modular group.

LEMMA 2. *If $xz^nx = z^i$ for some integers n, i, then $xz^n = z^nx$.*

PROOF. We have $z^i = xz^nx = (xzx)^n = (z^{-1}xz^{-1})^n$. Conjugation by z gives $z^i = (xz^{-2})^n$. Now conjugating by z^{-2}, we see that $z^i = (z^{-2}x)^n$. Then

$$xz^ix = x(z^{-2}x)^nx = (xz^{-2})^n = z^i = xz^nx.$$

Hence $z^i = z^n$ and $xz^n = z^nx$.

Let X' be another bordered Klein surface with maximal symmetry. Our first result establishes a relationship between fully wound coverings φ: $X \to X'$ and the structure of the automorphism group $G = A(X)$.

THEOREM 1. *Let φ: $X \to X'$ be a fully wound covering of a surface X' with maximal symmetry. If $G' = A(X')$ has index m, then in the group $G = A(X)$, $xz^m = z^mx$, and $X' = X/M$ where $M = \langle z^m \rangle$. Further m divides q, the index of G.*

PROOF. Let L be the group of covering transformations, so that $X' = X/L$ and $G' = G/L$. Let C be a component of ∂X fixed by the subgroup $\langle u, z \rangle$. The cyclic group L fixes C and φ is unramified, so that clearly $L \subset \langle z \rangle$. Let l be the least positive integer such that z^l generates L. Then l divides q. Now $xLx = L$ since L is normal in G. Therefore $xz^lx = z^i$ for some integer i, and by Lemma 2 $xz^l = z^lx$.

Now $o(G') = o(G)/o(L) = 2qk/(q/l) = 2lk$. But $o(G') = 2mk$ since G' has index m and X' has k boundary components. Thus $l = m$ and $L = \langle z^m \rangle = M$.

Wilson has obtained a similar result about coverings of regular maps [10].

Our next result will enable us to exhibit a surface with maximal symmetry as a fully wound covering of a surface with maximal symmetry of lower genus. The following lemma deals with a very special case. We continue the above notation.

LEMMA 3. *If* $q = 2b$ *and* $xz^b x = uz^b$, *then* $q = 4$, $G \cong S_4$, *and* X *is the projective plane with three holes.*

PROOF. We have

$$xuz = xuz^b z^b z = x(xz^b x)z^b z = z^b xzz^b.$$

Hence $o(xuz) = o(xz) = 3$. Returning to the generators $t = xu$, u, $v = uz$, we see that $o(tuv) = 3$. By Lemma 1, G also has index 3. Now G is determined; $G \cong S_4$ and $q = 4$ (see [**1**, pp. 37, 138] where the symbol [3, 3] is used for S_4). Finally, since $2qk = o(G) = 24$, $k = 3$ and X is the projective plane with three holes, the nonorientable surface with primitive maximal symmetry of genus 3.

THEOREM 2. *Let* X *be a bordered Klein surface with maximal symmetry, topologically different from the projective plane with three holes. Let* k *be the number of components of* ∂X, *and let* $G = A(X)$ *have index* q *and generators* u, x, z *satisfying the relations* (1.2). *Then there is a positive integer* $n \leqslant k$ *such that* n *divides* q *and* $xz^n = z^n x$. *Further the subgroup* $N = \langle z^n \rangle$ *is normal in* G.

PROOF. Let $S = \langle u, z \rangle$. Since $[G\colon S] = k$, the $k + 1$ cosets $(xzx)^m S$, $0 \leqslant m \leqslant k$, cannot all be distinct. Let n be the least positive integer such that $n \leqslant k$ and $(xzx)^n = xz^n x \in S$. Since $xz^q x \in S$, clearly n divides q. Now $xz^n x$ is an element of the dihedral group $S = \langle u, z \rangle$. Thus either $xz^n x = z^i$ or $xz^n x = uz^i$ for some integer i, $0 \leqslant i \leqslant q - 1$.

Suppose first that $xz^n x = uz^i$. Since $o(uz^i) = 2$, we see that $z^{2n} = 1$, so that either $n = q$ or $n = q/2$. It is not hard to see that all possibilities except $n = i = q/2$ produce a relation that causes collapse in the M^*-group G, that is, adding the new relation to the relations (1.2) results in a group of order smaller than 12. Thus $i = n = q/2$ and $xz^n x = uz^n$. By Lemma 3, X is the projective plane with three holes, contradicting the hypothesis.

Therefore $xz^n x = z^i$. Then $xz^n = z^n x$ by Lemma 2. Obviously $zNz^{-1} = N$ and $xNx = N$. But $uzu = z^{-1}$ so that $uz^n u = z^{-n}$ and $uNu = N$. Hence N is normal in G.

COROLLARY. *Let* $X' = X/N$ *and* $G' = G/N$. *If* $[G\colon N] > 6$, *then* X' *has maximal symmetry*, G' *has index* n, *and the quotient map* $\pi\colon X \to X'$ *is a fully wound covering.*

PROOF. By Theorem B, X' has maximal symmetry and π is a full covering. Since $N \subset S$ and S fixes a component of ∂X, it is clear that the boundary degree of π is $o(N)$. Thus π is fully wound. Since X' has k boundary components and $o(G') = 2qk/(q/n) = 2nk$, the M^*-group G' has index n.

REMARK. The only surfaces with maximal symmetry to which the corollary does not apply are the torus with one hole, a surface of genus 2, and of course the projective plane with three holes, a surface of genus 3. This can be seen by checking the surfaces with $k \leqslant 3$, since $[G\colon N] = 2nk > 6$ if $k > 3$.

If $k < q$, then we are assured of a nontrivial covering $\varphi\colon X \to X'$. This observation leads to the following result about surfaces with primitive maximal symmetry.

THEOREM 3. *Let X be a surface of genus $g > 3$ with k boundary components that has primitive maximal symmetry, and let $G = A(X)$ have index q. Then $q \leqslant k$ and $o(G) \leqslant 2k^2$.*

PROOF. We know that G/N is an M^*-group. Since G is M^*-simple, this means N is trivial and $n = q$. Thus $q \leqslant k$ and $o(G) = 2qk \leqslant 2k^2$.

Next we obtain a general bound for $o(G)$ in terms of k. We continue the above notation, and let $H = \langle x, z \rangle$. H is a subgroup of G of index at most two.

THEOREM 4. *Let X be a Klein surface with k boundary components that has maximal symmetry, and let $G = A(X)$. If $k > 1$, then $o(G) \leqslant 2k^3$.*

PROOF. First note that the bound holds for the nonorientable surface of genus three with $k = 3$. Thus we may assume that X is not this surface. Then since $k > 1$, the corollary to Theorem 2 applies. Hence G/N is an M^*-group with index n, and $[G\colon N] = 2nk \leqslant 2k^2$. We need to bound $o(N)$ in terms of k. We consider two cases, depending upon whether $G = H$ or $[G\colon H] = 2$. Note that N is central in H.

First suppose $G = H$. Then N is central in the big group G. In particular, $uz^n = z^n u$. Then $z^n = uz^n u = z^{-n}$ and $z^{2n} = 1$. Hence $o(N) \leqslant 2$ and $o(G) \leqslant 4k^2 \leqslant 2k^3$ since $2 \leqslant k$.

Now suppose $[G\colon H] = 2$. Since G/N is an M^*-group, 12 divides $o(G/N)$ so that 6 divides nk. We have the following chain of subgroups of G:

$$1 - N \overset{n}{-} \langle z \rangle \overset{k}{-} H \overset{2}{-} G.$$

Since N is central in H and $[H\colon N] = nk$, the mapping $\tau(h) = h^{nk}$ defines a homomorphism of H into N. This can be seen by considering the transfer of H into N [**2**, p. 45]. But H is generated by x and xz, elements of orders 2 and 3. Since 6 divides nk, the homomorphism τ takes x and xz to the identity. Therefore $\tau(z) = z^{nk} = 1$ and q divides nk. Then $o(N) = q/n \leqslant k$ and again we have $o(G) \leqslant 2k^3$.

We do not know of a surface with maximal symmetry for which the bound of Theorem 4 is attained, and perhaps the bound is a little generous. But for example there is a surface of genus 9 with $k = 4$ having maximal symmetry [**3**, §4]; for this surface $o(G) = 96 = 3k^3/2$.

Theorem 4 does however let us establish our main result.

THEOREM 5. *For each value of k there are only finitely many topological types of bordered Klein surfaces with maximal symmetry that have exactly k boundary components.*

PROOF. Fix the positive integer k. Let X be a surface of genus g with k boundary components that has maximal symmetry. If $k = 1$, then X is a torus with one hole [**3**, §2]. If $k > 1$, then $12(g - 1) \leqslant 2k^3$ and $g \leqslant 1 + k^3/6$. Thus there are only a finite number of possibilities for the genus g, and for each value of g there are at most two topological types of surfaces with k boundary components.

Theorem 4 also has the following interesting consequence.

THEOREM 6. *Let φ: $X \to Y$ be a fully wound covering of a surface Y with maximal symmetry. If Y has k boundary components, then the degree of the covering φ is at most k.*

PROOF. We may assume that X does not have primitive maximal symmetry. In particular, $k > 1$ and X is not the projective plane with 3 holes.

Let $G = A(X)$ have index q and generators u, x, z satisfying the relations (1.2). Let $G' = A(Y)$ have index m. Then by Theorem 1, m divides q, $xz^m = z^m x$, and $Y = X/M$ where $M = \langle z^m \rangle$. Using Theorem 2 let n be the least positive integer such that n divides q and $xz^n = z^n x$, and set $N = \langle z^n \rangle$. Clearly n divides m, so that $M \subset N$. Hence $o(M) \leqslant o(N) \leqslant k$ by the proof of Theorem 4. In fact the following diagram commutes.

$$\begin{tikzcd} & X \arrow[dl] \arrow[d] \\ X/M \arrow[r] & X/N \end{tikzcd}$$

It is somewhat surprising that the degree of a fully wound covering of a surface with maximal symmetry should be limited by the number of boundary components. There are fully wound coverings with the degree equal to the number of boundary components [3, §4]. For example, the torus with three holes, a surface of genus 4, is a 3-sheeted full covering of the sphere with three holes, a surface of genus 2.

4. Surfaces with a prime number of boundary components. Now we consider the problem of determining the possible values for the number of boundary components of a surface with maximal symmetry. For example, there are surfaces with maximal symmetry with k boundary components for $k = 1, 3, 4, 6, 9, 12$ and in fact for any k of the form n^2 or $3n^2$ [3]. Here we examine the surfaces with a prime number of boundary components. From the beginning we felt that it should be difficult for such a surface to have a large number of automorphisms. Again we use the concept of a fully wound covering, but we shall also need the following group-theoretic result.

LEMMA 4. *There are no M^*-groups of order $12p$ for any prime $p > 5$.*

PROOF. Let G be a group of order $12p$, where p is a prime larger than 5, and let S be a Sylow p-subgroup of G. Then S is normal in G. If $p \neq 11$, this follows immediately from the Sylow theorems. If $p = 11$, then G is solvable, and this follows from the extended Sylow theorems for solvable groups [4, p. 141]. Let C be the centralizer of S in G. Then G/C is isomorphic to a subgroup of the automorphism group of S [9, p. 50], that is, to a subgroup of a cyclic group. Therefore the commutator subgroup $G' \subset C$.

Now assume that G is an M^*-group. Then $[G: G']$ divides 4 and $[G': G'']$ divides 9 [3, §5]. Then S is the Sylow p-subgroup of G', and S is central in G'. By a theorem of Burnside [9, p. 137], S has a normal complement in G'. Then S is a factor group of G' and p divides $[G'\colon G'']$, a contradiction. Hence G is not an M^*-group.

Lemma 4 provides a purely algebraic proof that there are infinitely many values of the genus g for which there is no surface with maximal symmetry [7, p. 404].

THEOREM 7. *There are no bordered Klein surfaces with maximal symmetry that have exactly p boundary components for any prime $p \geqslant 5$.*

PROOF. Let X be a surface with p boundary components, where p is a prime, $p \geqslant 5$. Assume that X has maximal symmetry, and let $G = A(X)$. By Theorem 2 and its corollary, G has a normal subgroup N such that $Y = X/N$ has maximal symmetry, the M^*-group $Q = G/N$ has index $n \leqslant p$, and the quotient map π: $X \to Y$ is a fully wound covering. Then Y has p boundary components, and $o(Q) = 2np \leqslant 2p^2$.

Let S be a Sylow p-subgroup of Q. Since 12 divides $o(Q)$, we see that $o(S) = p$ and in fact $o(Q) < 2p^2$. Also note that $p \neq 5$. Let n_p be the number of Sylow p-subgroups of Q, and let $N(S)$ and $C(S)$ be the normalizer and centralizer of S in Q. Write $o(C(S)) = pr$ and $o(N(S)) = prw$. We have $o(Q) = prwn_p < 2p^2$ and $n_p \equiv 1 \bmod p$. Thus either $n_p = 1$ or $n_p = p + 1$. But if $n_p = p + 1$, then $r = w = 1$ and $N(S) = C(S)$. Then S would have a normal complement in Q [9, p. 137], and p would divide $[Q\colon Q']$, a contradiction. Therefore $n_p = 1$ and S is normal in Q.

Now $[Q\colon S] \geqslant 12$ since 12 divides $o(Q)$. Let $Z = Y/S$, $A = Q/S$, and let φ: $Y \to Z$ be the quotient map. By Theorem B, Z has maximal symmetry and φ is a full covering of degree p. Then the boundary degree d of φ divides p. Since Q has index n and n and p are relatively prime, it is clear from Theorem 1 that $d = 1$. Then by (2.1) Z has one boundary component. Hence Z is a surface of genus 2 [3, §2], $o(A) = 12$, and $o(Q) = 12p$. Since $p > 5$, this contradicts Lemma 4. Therefore the surface X does not have maximal symmetry.

Note the importance of Theorem 2 in the proof. Theorem 2 tells us that if there were a surface X with p boundary components that had maximal symmetry, then X would be a full cover of another such surface whose automorphism group is small in terms of p.

Finally we give the consequence of Theorem 7 for regular maps.

COROLLARY. *There are no regular maps of type $\{3, q\}$ that have exactly p vertices for any prime $p \geqslant 5$.*

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