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AN EXAMPLE OF A LIMINAL C^*-ALGEBRA

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ABSTRACT. For each countable ordinal γ there exists a unital separable liminal C^*-algebra A_γ with the property that if $(I_\rho)_{\rho=1}^{\beta}$ is any composition sequence of A_γ such that the spectra of the quotients $I_{\rho+1}/I_\rho$ are Hausdorff, then $\beta > \gamma + 1$. Moreover, there *is* a composition sequence $(I_\rho)_{\rho=1}^{\gamma+1}$ of A_γ such that the spectra of the quotients $I_{\rho+1}/I_\rho$ are Hausdorff.

1. Introduction. In [**2**, 4.7.25] J. Dixmier posed the following problem: Construct an example of a liminal C^*-algebra which does not admit any finite composition sequence (I_ρ) such that the spectra of the quotients $I_{\rho+1}/I_\rho$ are Hausdorff. Blackadar solved this problem in [**1**, p. 325]. The purpose of this note is to show that there are examples with the above property with respect to any countable ordinal. Specifically, we prove the following theorem.

THEOREM. *For each countable ordinal γ there exists a unital separable liminal C^*-algebra A_γ with the property that if $(I_\rho)_{\rho=1}^{\beta}$ is any composition sequence of A_γ such that the spectra of the quotients $I_{\rho+1}/I_\rho$ are Hausdorff, then $\beta > \gamma + 1$. Moreover, there is a composition sequence $(I_\rho)_{\rho=1}^{\gamma+1}$ of A_γ such that the spectra of the quotients $I_{\rho+1}/I_\rho$ are Hausdorff.*

2. The two-point T_1 compactification. In this section we present some topological results that are necessary for the proof of our theorem.

Let X be a topological space that is at least T_1 and let ∞_1, ∞_2 be points not in X. Let X_1 be the set $X \cup \{\infty_1, \infty_2\}$ together with the topology whose members are all subsets of U of X, such that (i) if $U \cap \{\infty_1, \infty_2\} = \varnothing$, then U is an open subset of X, (ii) if $U \cap \{\infty_1, \infty_2\} \neq \varnothing$, then the complement of $U \cup \{\infty_1, \infty_2\}$ is a closed compact subset of X. Clearly, X_1 is a compact T_1 space and if X is not compact, then X_1 is not Hausdorff. We call X_1 the two-point T_1 compactification of X.

Next suppose β is an ordinal. A topological space X is said to have a Hausdorff decomposition of length β if there exists a generalized sequence $\{U_\rho\}_{\rho=1}^{\beta}$ of open subsets of X with the following properties. (i) $U_\rho \subseteq U_{\rho+1}$, $\rho < \beta$, (ii) U_1 and $U_{\rho+1} \setminus U_\rho$, $\rho < \gamma$, are Hausdorff, (iii) if $\rho \leqslant \beta$ is a limit ordinal, then $U_\rho = \bigcup_{\delta<\rho} U_\delta$, (iv) $U_\beta = X$.

Now assume X is a denumerable discrete set and let X_1 denote the two-point T_1 compactification of X. Next let β be a countable ordinal. Suppose that for all $\rho < \beta$, X_ρ has already been defined. If β is not a limit ordinal, let X_β be the two-point T_1 compactification of the disjoint union $\bigcup_{n=1}^{\infty} Y_n$, where each Y_n is a

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copy of $X_{\beta-1}$. If β is a limit ordinal, let X_β be the two-point T_1 compactification of the disjoint union $\bigcup_{\rho<\beta} X_\rho$. We have defined by induction a compact T_1 topological space X_β for each countable ordinal β.

2.1. LEMMA. *Let β be a countable ordinal and let X_β be the topological space defined above. Then X_β does not admit any Hausdorff decomposition of length less than or equal to β but does admit one of length $\beta + 1$.*

PROOF. Clearly the assertion holds for the topological space X_1. Now suppose the assertion holds for all topological spaces X_ρ, $\rho < \beta$. First, assume β is not a limit ordinal. Recall that X_β is the two-point T_1 compactification of the disjoint union $\bigcup_{n=1}^\infty Y_n$, where each Y_n is a copy of $X_{\beta-1}$. Let $\{V_\rho\}_{\rho=1}^\beta$ be a Hausdorff decomposition of $X_{\beta-1}$ of length β. Set $U_{\beta+1} = X_\beta$ and $U_\rho = \bigcup_{n=1}^\infty V_{\rho,n}$, $\rho \leqslant \beta$, where $V_{\rho,n}$ is the copy of V_ρ in Y_n. Clearly, $\{U_\rho\}_{\rho=1}^{\beta+1}$ is a Hausdorff decomposition of X_β of length $\beta + 1$. Now suppose $\{U_\rho\}_{\rho=1}^\gamma$ is a Hausdorff decomposition of X_β of length γ. Assume $\gamma \leqslant \beta$. If $\gamma < \beta$, then $\{U_\rho \cap Y_1\}_{\rho=1}^\gamma$ is a Hausdorff decomposition of $X_{\beta-1}$ of length less than or equal to $\beta - 1$, a contradiction. So assume $\gamma = \beta$. If $Y_n \setminus U_{\beta-1} \neq \varnothing$, $n = 1, 2, 3, \ldots$, then there is a sequence in the complement of $U_{\beta-1}$ that converges to two points in the complement of $U_{\beta-1}$. This contradicts the fact that $U_\beta \setminus U_{\beta-1}$ is Hausdorff decomposition. Thus there is a $Y_n \subset U_{\beta-1}$. But this implies that $\{U_\rho \cap Y_n\}_{\rho=1}^{\beta-1}$ is a Hausdorff decomposition of Y_n of length $\beta - 1$, another contradiction. Thus $\gamma \geqslant \beta + 1$.

Suppose β is a limit ordinal. Recall that X_β is the two-point T_1 compactification of the disjoint union $\bigcup_{\rho<\beta} X_\rho$. For each $\rho < \beta$ let $\{V_{\rho,\delta}\}_{\delta=1}^{\rho+1}$ be a Hausdorff decomposition of X_ρ of length $\rho + 1$. Set $U_1 = \bigcup_{\rho=1}^{\beta} V_{\rho,1}$ and let δ be an ordinal such that $1 < \delta \leqslant \beta$. Assume that U_α has been defined for all $\alpha < \delta$. If δ is not a limit ordinal, then set

$$U_\delta = \left(\bigcup_{\delta<\alpha<\beta} V_{\alpha,\delta}\right) \cup \left(\bigcup_{\alpha<\delta} X_\alpha\right).$$

If δ is a limit ordinal, then set $U_\delta = \bigcup_{\alpha<\delta} U_\alpha$. Next set $U_{\beta+1} = X_\beta$. Clearly $\{U_\rho\}_{\rho=1}^{\beta+1}$ is a Hausdorff decomposition of X_β of length $\beta + 1$. Finally suppose $\{U_\rho\}_{\rho=1}^\gamma$ is a Hausdorff decomposition of X_β of length γ. Then $\{U_\rho \cap X_\alpha\}_{\rho=1}^\gamma$ is a Hausdorff decomposition of X_α of length greater than or equal to $\alpha + 1$ for all $\alpha < \beta$. This implies $\gamma \geqslant \alpha + 1$ for all $\alpha < \beta$. Thus $\gamma \geqslant \beta$. If $\beta = \gamma$, then $X_\beta = U_\beta = \bigcup_{\rho<\beta} U_\rho$. Since X_β is compact, there exists a $\rho < \beta$ such that $X_\beta = U_\rho$. This means $\{U_\delta \cap X_\rho\}_{\delta=1}^\rho$ is a Hausdorff decomposition of X_ρ of length ρ, which contradicts our induction hypothesis. Thus our proof is complete.

3. The example. For each positive integer n let A_n denote a C^*-algebra with identity I_n and T_1 spectrum $\hat{A}_n$. Let p_n be a nontrivial projection in A_n such that $\pi(p_n) \neq 0 \neq \pi(I_n - p_n)$ for all $\pi \in \hat{A}_n$; set $q_n = I_n - p_n$. Let $\Sigma \oplus A_n$ denote the direct sum of the sequence $\{A_n\}$ and let $\mathcal{Q}(\{(A_n, p_n)\})$ denote the set of all $x \in \Sigma \oplus A_n$ with the following property. There exist complex numbers λ_1 and λ_2

such that

$$\lim_{n\to\infty} \|x(n) - \lambda_1 p_n - \lambda_2 q_n\| = 0.$$

3.1. PROPOSITION. *Let $\mathcal{Q}(\{(A_n, p_n)\})$ be defined as above. Then $\mathcal{Q} = \mathcal{Q}(\{(A_n, p_n)\})$ is a unital C^*-algebra whose spectrum $\hat{\mathcal{Q}}$ is the two-point T_1 compactification of the disjoint union $\bigcup_{n=1}^{\infty} \hat{A}_n$. Moreover if each A_n is separable and liminal, then $\mathcal{Q}$ is separable and liminal.*

PROOF. It is straightforward to verify that $\mathcal{Q}$ is a unital C^*-algebra and if each A_n is separable, it is clear that $\mathcal{Q}$ is separable. Now let T be the two-point T_1 compactification of the disjoint union $\bigcup_{n=1}^{\infty} \hat{A}_n$ and let π_1, π_2 be its points at infinity. Define the map Ψ: $T \to \hat{\mathcal{Q}}$ by the following formula. For $\pi_i \in T$, $i = 1, 2$, and $x \in \mathcal{Q}$ let $\Psi(\pi_i)(x) = \lambda_i$, where λ_1 and λ_2 are the unique complex numbers for which

$$\lim_{n\to\infty} \|x(n) - \lambda_1 p_n - \lambda_2 q_n\| = 0.$$

Clearly $\Psi(\pi_i)$, $i = 1, 2$, belong to $\hat{\mathcal{Q}}$. If $\pi \in A_n$ for some positive integer n and $x \in \mathcal{Q}$, define $\Psi(\pi)(x) = \pi(x(n))$. Clearly Ψ is a one-to-one map of T into $\hat{\mathcal{Q}}$. Now we wish to show that Ψ is an onto map. Let J_{π_1} be the set of all x in $\mathcal{Q}$ for which there is a complex number λ such that $\|x(n) - 0 \cdot p_n - \lambda q_n\| \to 0$. Similarly define J_{π_2}. Let $p, q \in \mathcal{Q}$ be defined by $p(n) = p_n$, $q(n) = q_n$. Note $p\mathcal{Q} \subseteq J_{\pi_2}$ and $q\mathcal{Q} \subseteq J_{\pi_1}$. Thus it easily follows that J_{π_i}, $i = 1, 2$, are closed two-sided ideals of $\mathcal{Q}$ such that $J_{\pi_1} + J_{\pi_2} = \mathcal{Q}$. Let ρ: $\mathcal{Q} \to B(H)$ be an irreducible representation of $\mathcal{Q}$. Without loss of generality we may assume $\rho|J_{\pi_1}$ is nondegenerate. If $\rho|J_n = \pi$ is nondegenerate for some integer n, where J_n is defined to be the set of all x in $\mathcal{Q}$ such that $x(m) = 0$, $m \neq n$, then by virtue of [2, 2.10.4], $\Psi(\pi) = \rho$. So assume $\rho|J_n = 0$ for $n = 1, 2, \ldots$. For each positive integer n define $e_n \in J_{\pi_1}$ by the formula

$$e_n(m) = \begin{cases} I_m, & m \leq n, \\ q_m, & m > n. \end{cases}$$

Clearly, $\{e_n\}$ is an approximate identity for J_{π_1}. Next let $x \in J_{\pi_1}$ and λ be the unique complex number for which

$$\|x(n) - 0 \cdot p_n - \lambda q_n\| \to 0.$$

By the assumption made on ρ, $\rho(x - \lambda e_m) \to 0$. So for $h \in H$, $\rho(x)(h) = \lim_{m\to\infty} \rho(x - \lambda e_m)(h) + \lambda\rho(e_m)(h) = \lambda I_H(h)$ by virtue of [2, 2.2.10]. In particular, $\rho(q) = I_H$. Next let $x \in \mathcal{Q}$ and λ_1, λ_2 be the unique complex numbers for which $\|x(n) - \lambda_1 p_n - \lambda_2 q_n\| \to 0$. It follows that $\rho(x)(h) = \rho(x)\rho(q)(h) = \rho(xq)(h) = \lambda_2 h$, that is, $\rho = \Psi(\pi_2)$. We have now proved that Ψ maps T onto $\hat{\mathcal{Q}}$. This proves, by the way, that if each A_n is liminal, then $\mathcal{Q}$ is liminal.

By virtue of [2, 3.2.1] Ψ is a homeomorphism of $\hat{A}_n$ onto $\hat{\mathcal{Q}}^{J_n}$ and by [2, 3.2.3] $\hat{\mathcal{Q}}^{J_n}$ is an open and closed compact subset of $\hat{\mathcal{Q}}$. Thus Ψ is a homeomorphism of the disjoint union $\bigcup \hat{A}_n$ onto $\bigcup \hat{\mathcal{Q}}^{J_n}$. It remains to be shown that the family of open neighborhoods of π_i is mapped onto the family of open neighborhoods of $\rho_i = \Psi(\pi_i)$, $i = 1, 2$. Let U be an open subset of $\hat{\mathcal{Q}}$ containing ρ_1. Then there is a closed ideal J of A such that $U = \hat{\mathcal{Q}}^J$. Thus there is an $x \in J$ such that $\rho_1(x) \neq 0$. Let λ_1

and λ_2 be the unique complex numbers for which

$$\|x(n) - \lambda_1 p_n - \lambda_2 q_n\| \to 0.$$

To say $\rho_1(x) \neq 0$ means $\lambda_1 \neq 0$. So xp has the property $\rho_1(xp) = \lambda_1 \neq 0$. Now suppose there is a subsequence $\{n_k\}$ of positive integers and elements $\pi_k \in \hat{A}_{n_k}$ for which $\Psi(\pi_k) \notin U$. It follows that

$$0 = \Psi(\pi_k)(xp) = \pi_k(xp(n_k)) = \pi_k(xp(n_k) - \lambda_1 p(n_k)) + \lambda_1 \pi_k(p(n_k)).$$

Since $\pi_k(p(n_k)) \neq 0$, $\|\pi_k(p(n_k))\| = 1$. Thus

$$|\lambda_1| = \|\pi_k(x(n_k) - \lambda_1 p_{n_k} - \lambda_2 q_{n_k})p_{n_k}\| \to 0,$$

a contradiction. So there exists a positive integer N such that $\Psi(\hat{A}_n) \subset U$ for all $n > N$. Now it is straightforward to verify that

$$\Psi^{-1}(U) \cup \{\pi_1, \pi_2\} = \left(\bigcup_{n=1}^{N} \Psi^{-1}(\hat{\mathcal{Q}}^{J_n} \cap U) \right) \cup \left(\bigcup_{n=N+1}^{\infty} \hat{A}_n \right) \cup \{\pi_1, \pi_2\}$$

so clearly $\Psi^{-1}(U)$ is an open neighborhood of π_1. Similarly, $\Psi^{-1}(U)$ is an open neighborhood of π_2 whenever U is an open neighborhood of ρ_2.

Finally, let V be an open subset of T containing π_1. There is a positive integer N such that for $n \geqslant N$ we have $\hat{A}_n \subseteq V$. Set $V_1 = \cup_{n=1}^{N-1}(V \cap \hat{A}_n)$ and $V_2 = V \setminus V_1$. Clearly, V_1 and V_2 are open. Moreover, it is clear that $\Psi(V_1)$ is open in $\hat{\mathcal{Q}}$. If $\pi_2 \in V_2$, put $J = \{x \in \mathcal{Q}\colon x(n) = 0,\ n < N\}$. If $\pi_2 \notin V_2$, put $J = J_{\pi_2} \cap \{x \in \mathcal{Q}\colon x(n) = 0,\ n < N\}$. It is easy to see that in either case J is a closed two-sided ideal of $\mathcal{Q}$ and that $\Psi(V_2) = \hat{\mathcal{Q}}^J$. So $\Psi(V) = \Psi(V_1) \cup \Psi(V_2)$ is an open neighborhood of ρ_1. Similarly $\Psi(V)$ is an open neighborhood of ρ_2 whenever V is an open neighborhood of π_2. This completes the proof.

Remark. Let $\{(B_n, p_n')\}$ be any rearrangement of the sequence $\{(A_n, p_n)\}$. The above result clearly implies that the spectrum of $\mathcal{Q}(\{(A_n, p_n)\})$ is homeomorphic to the spectrum of $\mathcal{Q}(\{(B_n, p_n')\})$. Actually a stronger statement can be made. The C^*-algebra $\mathcal{Q}(\{(A_n, p_n)\})$ is $*$-isomorphic to the C^*-algebra $\mathcal{Q}(\{(B_n, p_n')\})$. Of course, if each A_n is separable then, by [2, 9.5.3], each A_n is liminal since it has a T_1 spectrum.

3.2. Proof of the Theorem. Let A_1 be the C^*-algebra of all sequences of 2×2 matrices of complex numbers which converge to a diagonal matrix. By [2, 4.7.19] A_1 is a unital separable liminal C^*-algebra whose spectrum is homeomorphic to the space X_1 defined in Lemma 2.1. Let γ be a countable ordinal and suppose, for each $\rho < \gamma$, A_ρ is a unital separable liminal C^*-algebra whose spectrum is homeomorphic to the space X_ρ defined in Lemma 2.1. First, assume γ is not a limit ordinal. For each positive integer n, let $M_2(A_{\gamma-1})$ denote the C^*-algebra of 2×2 matrices over $A_{\gamma-1}$. From [3, Corollaire 5, Lemme 16] we see that $M_2(A_{\gamma-1})$ is a unital separable liminal C^*-algebra whose spectrum $M_2^\wedge(A_{\gamma-1})$ is homeomorphic to $\hat{A}_{\gamma-1}$, the spectrum of $A_{\gamma-1}$. Thus $M_2^\wedge(A_{\gamma-1})$ is homeomorphic to $X_{\gamma-1}$. For each positive integer n let $B_n = M_2(A_{\gamma-1})$ and

$$p_n = \begin{bmatrix} I_{\gamma-1} & 0 \\ 0 & 0 \end{bmatrix},$$

where $I_{\gamma-1}$ is the identity of $A_{\gamma-1}$. Now set $A_\gamma = \mathcal{Q}(\{(B_n, p_n)\})$ as defined in Proposition 3.1. Thus by Proposition 3.1 and our induction hypothesis A_γ is a unital separable liminal C^*-algebra whose spectrum $\hat{A}_\gamma$ is homeomorphic to X_γ. Now assume γ is a limit ordinal. Since γ is countable, the set of all $\rho < \gamma$ can be put in a sequence $\{\rho_n\}_{n=1}^{\infty}$. For each positive integer n, let $B_n = M_2(A_{\rho_n})$ and

$$p_n = \begin{bmatrix} I_{\rho_n} & 0 \\ 0 & 0 \end{bmatrix},$$

where I_{ρ_n} is the identity of A_{ρ_n}. Set $A_\gamma = \mathcal{Q}(\{(B_n, p_n)\})$ as defined in Proposition 3.1. Again by Proposition 3.1, [3] and our induction hypothesis, A_γ is a unital separable liminal C^*-algebra whose spectrum $\hat{A}_\gamma$ is homeomorphic to X_γ. The assertion of the theorem follows immediately from Lemma 2.1 and [2, 3.2.1].

References

1. B. Blackadar, *Infinite tensor products of C^*-algebras*, Pacific J. Math. **72** (1977), 313–334.
2. J. Dixmier, *Les C^*-algèbres et leur représentations*, Gauthier-Villars, Paris, 1964.
3. A. Wulfson, *Produit tensoriel de C^*-algèbres*, Bull. Sci. Math. **87** (1963), 13–28.

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