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LOCAL ERGODIC THEOREMS FOR NONCOMMUTING SEMIGROUPS

S. A. McGRATH

ABSTRACT. Let (X, μ) be a σ-finite measure space and $L_p(\mu)$, $1 < p < \infty$, the usual Banach spaces of complex-valued functions. For $k = 1, 2, \ldots, n$, let $\{T_k(t): t > 0\}$ be a strongly continuous semigroup of Dunford-Schwartz operators. If

$$f \in R_{n-1} = \left\{ g: \int_{|g|>t} |g/t|(\log|g/t|)^{n-1}\, d\mu < \infty \text{ for all } t > 0 \right\},$$

then

$$\frac{1}{\alpha_1\alpha_2 \cdots \alpha_n} \int_0^{\alpha_n} \cdots \int_0^{\alpha_1} T_n(t_n) \cdots T_1(t_1)f(x)\, dt_1 \cdots dt_n \to f(x)$$

μ-a.e. as $\alpha_1 \searrow 0, \ldots, \alpha_n \searrow 0$ independently. If $f \in L_p(\mu)$, $1 < p < \infty$, then the limit exists in norm as well as pointwise.

Introduction. Let (X, μ) be a complete, σ-finite measure space and let $L_p(\mu) = L_p(X, \mu)$, $1 < p < \infty$, be the usual Banach spaces of complex-valued functions. Let $\{T(t): t > 0\}$ be a strongly continuous semigroup of $L_1(\mu)$ contractions. This means that (i) $T(t + s) = T(t)T(s)$, $s, t > 0$; (ii) $\|T(t)\|_1 < 1$, $t > 0$; (iii) $f \in L_1(\mu)$ implies $\|T(t)f - T(s)f\|_1 \to 0$ as $s \to t$. We assume for simplicity that $T(0) = I$. A semigroup $\{T(t)\}$ of $L_1(\mu)$ contractions is a Dunford-Schwartz semigroup if $\|T(t)\|_\infty < 1$ for all $t > 0$. It is a submarkovian semigroup if each $T(t)$ is a positive operator, i.e. $f \in L_1^+(\mu)$ implies $T(t)f \in L_1^+(\mu)$ for all $t > 0$. A positive $L_1(\mu)$ semigroup $\{P(t)\}$ is said to dominate $\{T(t)\}$ if $P(t)|f| > |T(t)f|$ μ-a.e. for $f \in L_1(\mu)$ and $t > 0$.

The strong continuity of $\{T(t)\}$ permits us to define, for $\alpha > 0$ and $f \in L_1(\mu)$, the integral $\int_0^\alpha T(t)f\,dt$ as the L_1-limit of Riemann sums. A more precise definition of $\int_0^\alpha T(t)f\,dt$ is required to investigate the pointwise convergence of $(1/\alpha)\int_0^\alpha T(t)f\,dt$. It is well known ([2], [8]) that given $f \in L_1(\mu)$ the vector $T(t)f$ has a scalar representation $T(t)f(x)$, defined on $R^+ \times X$ and measurable with respect to the product measure on $R^+ \times X$, such that $T(t)f(x)$ is in the equivalence class of $T(t)f$ for all $t > 0$. This representation is unique modulo sets of product measure zero. The scalar function $T(t)f(x)$ is integrable with respect to the product measure on $R^+ \times X$. Additionally, there is a μ-null set $E(f)$, independent of $\alpha > 0$, outside which $\int_0^\alpha T(t)f(x)\,dt$ exists and, as a function of x, is in the equivalence class of $\int_0^\alpha T(t)f\,dt$ for every $\alpha > 0$. We define

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$$A(T, \alpha)f(x) = \frac{1}{\alpha}\int_0^\alpha T(t)f(x)\,dt$$

for all $\alpha > 0$ and $f \in L_1(\mu)$.

In [3] N. Fava showed that if $f \in R_{n-1}$ and $\{T_k(t): t \geq 0\}$, $k = 1, 2, \ldots, n$, are strongly continuous semigroups of positive Dunford-Schwartz operators, then

$$\frac{1}{\alpha_1\alpha_2 \cdots \alpha_n}\int_0^{\alpha_n} \cdots \int_0^{\alpha_1} T_n(t_n) \cdots T_1(t_1)f(x)\,dt_1 \cdots dt_n$$

converges μ-a.e. to a finite limit as $\alpha_1 \to \infty, \ldots, \alpha_n \to \infty$ independently. The class

$$R_n = \left\{ f: \int_{|f|>t} |f/t|(\log|f/t|)^n\,d\mu < \infty \text{ for all } t > 0 \right\}$$

is a subspace of $L_1(\mu) + L_\infty(\mu)$ and satisfies $L_1(\mu) + L_\infty(\mu) \supset R_0 \supset R_1 \supset R_2 \supset \cdots$. Also, for any $1 < p < \infty$ and $n \geq 0$, $L_p(\mu) \subset R_n$. Finally, $R_n = L(\log^+ L)^n$, for all $n \geq 0$, when $\mu(X) < \infty$. These facts are established in [3].

In this note a local ergodic theorem is established: if $\{T_k(t)\}$, $k = 1, 2, \ldots, n$, are strongly continuous semigroups of Dunford-Schwartz operators and $f \in R_{n-1}$, then

$$\frac{1}{\alpha_1 \cdots \alpha_n}\int_0^{\alpha_n} \cdots \int_0^{\alpha_1} T_n(t_n) \cdots T_1(t_1)f(x)dt_1 \cdots dt_n \to f(x) \quad \mu\text{-a.e.} \quad (*)$$

as $\alpha_1 \searrow 0, \ldots, \alpha_n \searrow 0$ independently. For notational convenience we denote the integral in (*) by $A(T_n, \alpha_n) \cdots A(T_1, \alpha_1)f(x)$. Local ergodic theorems for single L_1 contraction semigroups have been established in ([4], [5], [6], [8], [9]). T. Terrell [10] extended the local ergodic theorem for one-parameter submarkovian semigroups to the n-parameter case. He showed that if $f \in L_1(\mu)$ then

$$\lim_{\alpha \searrow 0} (1/\alpha)^n \int_0^\alpha \cdots \int_0^\alpha T(t_1, \ldots, t_n)f(x)dt_1 \cdots dt_n = f(x) \quad \mu\text{-a.e.}$$

He pointed out that if it is assumed only that $f \in L_1(\mu)$ then (*) may fail (even if the semigroups commute).

Main results. If $\{T(t): t \geq 0\}$ is a strongly continuous $L_1(\mu)$ semigroup of Dunford-Schwartz operators then a scalar representation $T(t)f(x)$ exists for any $f \in L_p(\mu)$, $1 \leq p < \infty$ ([2, pp. 196–198]). However R_n is not contained in the linear span of $\bigcup_{1 \leq p < \infty} L_p(\mu)$ [3]. Before proving our ergodic theorem we must show that a scalar representation exists for functions in R_0.

1. Lemma. *Let (X, μ) be a complete, σ-finite measure space and let $\{T(t)\}$ be a strongly continuous $L_1(\mu)$ contraction semigroup such that for all $t \geq 0$, $\|T(t)f\|_\infty \leq \|f\|_\infty$, $f \in L_1(\mu) \cap L_\infty(\mu)$. Then $\{T(t)\}$ may be extended to a Dunford-Schwartz semigroup and the domain of definition of the scalar representation of $\{T(t)\}$ may be extended from $L_1(\mu)$ to $L_1(\mu) + L_\infty(\mu)$.*

Proof. Let $\{P(t): t \geq 0\}$ be a strongly continuous submarkovian semigroup which dominates $\{T(t): t \geq 0\}$ ([4], [6]). If $f \in L_1^+(\mu)$ then $P: f \to e^{-t}P(t)f(x)$ defines a linear contraction mapping from $L_1(\mu)$ to $L_1(R^+ \times X, d\rho)$, where $d\rho = dt \times d\mu$. This mapping is a positive contraction since if $f \geq 0$ μ-a.e. then $P(t)f \geq 0$

μ-a.e. for all $t \geq 0$ implies $Pf \geq 0$ ρ-a.e. by Fubini's theorem. Consequently P may be extended to the cone of positive measurable functions on X [7]. We denote the extension of P by $\hat{P}$.

Choose $f \in L_\infty^+(\mu)$ and let f_k $(k = 1, 2, \dots)$ be a sequence of functions in $L_1^+(\mu) \cap L_\infty^+(\mu)$ with $f_k(x) \nearrow f(x)$ μ-a.e. as $k \to \infty$. Then $\hat{P}f(t, x) = \lim_{k\to\infty} e^{-t}P(t)f_k(x)$, since $\hat{P}$ has the monotone continuity property. We have $|\hat{P}f(t, x)| < \infty$ ρ-a.e. since $\|\hat{P}g\|_\infty \leq \|g\|_\infty$ for all $g \in L_\infty^+(\mu)$. Consequently the sequence $\{T(t)f_k(x)\}$ is Cauchy μ-a.e. since

$$|T(t)f_{k+j}(x) - T(t)f_k(x)| \leq |P(t)f_{k+j}(x) - P(t)f_k(x)|.$$

Set $\hat{T}(t)f(x) = \lim_{k\to\infty} T(t)f_k(x)$, $f \in L_\infty^+(\mu)$. It is not difficult to show that our definition is independent, modulo ρ-null sets, of the particular sequence $\{f_k\}$ converging to f. Extend now $\hat{T}(t)f(x)$ to $L_\infty(\mu)$ by linearity. If we set $T(t)f = \hat{T}(t)f(\cdot)$, $f \in L_\infty(\mu)$, then we have extended $\{T(t)\}$ to a Dunford-Schwartz semigroup and $\hat{T}(t)f(x)$ is in the equivalence class of $T(t)f$ for all $t \geq 0$ and $f \in L_\infty(\mu)$. One can see that given $f \in L_\infty(\mu)$, there exists a μ-null set $E(f)$, independent of $\alpha > 0$, outside of which $\int_0^\alpha \hat{T}(t)f(x)\,dt$ exists and is finite. If $f \in L_1(\mu) \cap L_\infty(\mu)$ then $\hat{T}(t)f(x)$ and $T(t)f(x)$ are equivalent scalar representations of $T(t)f$. Finally, if $f = g + h$, where $g \in L_1(\mu)$ and $h \in L_\infty(\mu)$, we define

$$T(t)f(x) = \hat{T}(t)g(x) + \hat{T}(t)h(x).$$

We note that this definition of $T(t)f(x)$ is independent, modulo a ρ-null set, of the particular g and h chosen for the representation of f. $\square$

2. THEOREM. *Let (X, μ) be a complete, σ-finite measure space and let $\{T_k(t): t \geq 0\}$, $k = 1, 2, \dots, n$, be strongly continuous $L_1(\mu)$ contraction semigroups such that, for all $t \geq 0$ and $f \in L_1(\mu) \cap L_\infty(\mu)$, $\|T_k(t)f\|_\infty \leq \|f\|_\infty$. If $f \in L_p(\mu)$, $1 < p < \infty$, then $A(T_n, \alpha_n) \cdots A(T_1, \alpha_1)f(x) \to f(x)$ both in norm and pointwise as $\alpha_1 \searrow 0, \dots, \alpha_n \searrow 0$ independently.*

PROOF. By Lemma 1 each $\{T_k(t)\}$ may be regarded as a Dunford-Schwartz semigroup. As pointed out in [2], $\{T_k(t)\}$ is a strongly continuous semigroup of $L_p(\mu)$ contractions for $1 < p < \infty$. If $f \in L_p(\mu)$ and if $f_k^*(x)$ denotes $\sup_{\alpha>0}|(1/\alpha)\int_0^\alpha T_k(t)f(x)\,dt|$, then $\|f_k^*\|_p \leq (p/(p-1))\|f\|_p$ by Theorem VIII.7.7 in [2]. Since $L_1(\mu) \cap L_p(\mu)$ is dense in $L_p(\mu)$ and $\lim_{\alpha_k\searrow 0} A(T_k, \alpha_k)f(x) = f(x)$ μ-a.e. for all $f \in L_1(\mu) \cap L_p(\mu)$ by Ornstein's theorem [8, p. 108], it follows from Banach's convergence principle [2, Theorem IV.11.3] that, for each k, $\lim_{\alpha_k\searrow 0} A(T_k, \alpha_k)f(x)$ exists and is finite μ-a.e. as $\alpha_k \searrow 0$ through some countable set, say Q^+, the set of positive rationals. Since $A(T_k, \alpha_k)f(x)$ depends continuously on α_k for a.e. x, we have $\lim_{\alpha_k\searrow 0} A(T_k, \alpha_k)f(x)$ exists and is finite μ-a.e. for every $f \in L_p(\mu)$. The fact that $\lim_{\alpha_k\searrow 0} A(T_k, \alpha_k)f(x) = f(x)$ μ-a.e. follows from the strong continuity of $\{T(t)\}$ at $t = 0$.

Now consider the convergence of $A(T_n, \alpha_n) \cdots A(T_1, \alpha_1)f(x)$. Since

$$|A(T_n, \alpha_n) \cdots A(T_1, \alpha_1)f(x) - f(x)| \leq \sum_{k=1}^{n-1} |A(T_n, \alpha_n) \cdots A(T_{k+1}, \alpha_{k+1})[A(T_k, \alpha_k)f(x) - f(x)]| + |A(T_n, \alpha_n)f(x) - f(x)|,$$

our result is established if we can show

$$A(T_n, \alpha_n) \cdots A(T_{k+1}, \alpha_{k+1})[A(T_k, \alpha_k)f(x) - f(x)] \to 0 \qquad \mu\text{-a.e.}$$

for $k = 1, 2, \ldots, n-1$. If $\{P_1(t)\}, \ldots, \{P_k(t)\}$ are strongly continuous semigroups of positive Dunford-Schwartz operators which dominate, respectively, $\{T_1(t)\}, \ldots, \{T_n(t)\}$, then

$$|A(T_n, \alpha_n) \cdots A(T_{k+1}, \alpha_{k+1})[A(T_k, \alpha_k)f(x) - f(x)]| \leq A(P_n, \alpha_n) \cdots A(P_{k+1}, \alpha_{k+1})[|A(T_k, \alpha_k)f(x) - f(x)|].$$

We have

$$\|A(P_n, \alpha_n) \cdots A(P_{k+1}, \alpha_{k+1})[A(T_k, \alpha_k)f(x) - f(x)]\|_p \leq ((p-1)/p)^{n-k}\|A(T_k, \alpha_k)f(x) - f(x)\|_p, \qquad k = 1, 2, \ldots, n-1.$$

Consequently, given $\varepsilon > 0$,

$$\mu\left\{ \limsup_{\alpha_1 \searrow 0, \ldots, \alpha_n \searrow 0} |A(T_n, \alpha_n) \cdots A(T_{k+1}, \alpha_{k+1})[A(T_k, \alpha_k)f(x) - f(x)]| > \varepsilon \right\} \leq \left(\frac{1}{\varepsilon}\right)^p \left[\left(\frac{p}{p-1}\right)^p\right]^{n-k} \cdot \|A(T_k, \alpha_k)f(\cdot) - f\|_p^p \quad \text{for any } \alpha_k > 0.$$

Since $\|A(T_k, \alpha_k)f - f\|_p^p \to 0$ as $\alpha_k \searrow 0$ by dominated convergence, we have

$$A(T_n, \alpha_n) \cdots A(T_{k+1}, \alpha_{k+1})[A(T_k, \alpha_k)f(x) - f(x)] \to 0$$

μ-a.e. as $\alpha_1 \searrow 0, \ldots, \alpha_n \searrow 0$ independently. The norm convergence of $A(T_n, \alpha_n) \cdots A(T_1, \alpha_1)f$ follows from dominated convergence. □

3. **Theorem.** *Let $\{T_k(t)\}$, $k = 1, 2, \ldots, n$, be strongly continuous $L_1(\mu)$ contraction semigroups such that $\|T_k(t)f\|_\infty \leq \|f\|_\infty$, $f \in L_1(\mu) \cap L_\infty(\mu)$. Then*

$$\lim_{\alpha_1 \searrow 0, \ldots, \alpha_n \searrow 0} A(T_n, \alpha_n) \cdots A(T_1, \alpha_1)f(x) = f(x) \quad \mu\text{-a.e.}$$

for $f \in R_{n-1}$.

Proof. For $f \in R_{n-1}$, set

$$\omega(f) = \limsup_{\alpha_1 \searrow 0, \ldots, \alpha_n \searrow 0} A(T_n, \alpha_n) \cdots A(T_1, \alpha_1)f(x) - \liminf_{\alpha_1 \searrow 0, \ldots, \alpha_n \searrow 0} A(T_n, \alpha_n) \cdots A(T_1, \alpha_1)f(x).$$

Choose a sequence f_k $(k = 1, 2, \ldots)$ of functions in $L_p(\mu)$, $1 < p < \infty$, with $f_k(x) \to f(x)$ μ-a.e. and $|f - f_k| \leq |f|$ for all k. Then

$$\omega(f) \leq \omega(f - f_k) + \omega(f_k) \leq \omega(f - f_k)$$

by the preceding theorem. So $\omega(f) \leqslant \omega(f - f_k) \leqslant 2(f - f_k)^*$, where

$$(f - f_k)^*(x) = \sup_{\alpha_1 \searrow 0, \ldots, \alpha_n \searrow 0} \left| A(T_n, \alpha_n) \cdots A(T_1, \alpha_1)(f - f_k)(x) \right|.$$

Since the semigroups $\{T_1(t)\}, \ldots, \{T_n(t)\}$ are dominated by positive Dunford-Schwartz semigroups, it follows from Fava's dominated estimate [3] that

$$\begin{aligned} \mu\{\omega(f) > 8t\} &\leqslant \mu\{(f - f_k)^* > 4t\} \\ &\leqslant C_{n-1} \int_{|f - f_k| > t} \left| \frac{f - f_k}{t} \right| \left(\log \left| \frac{f - f_k}{t} \right| \right)^{n-1} d\mu \quad \text{for any } t > 0. \end{aligned}$$

The integral approaches zero as $k \to \infty$ by dominated convergence. Thus $\mu\{\omega(f) > 8t\} = 0$ for all $t > 0$, and so $\lim_{\alpha_1 \searrow 0, \ldots, \alpha_n \searrow 0} A(T_n, \alpha_n) A(T_{n-1}, \alpha_{n-1}) \cdots A(T_1, \alpha_1) f(x)$ exists and is finite μ-a.e. Setting

$$\tilde{f}(x) = \lim_{\alpha_1 \searrow 0, \ldots, \alpha_n \searrow 0} A(T_n, \alpha_n) \cdots A(T_1, \alpha_1) f(x),$$

we have

$$\begin{aligned} \mu\left\{ (\tilde{f} - f_k) > 4t \right\} &= \mu\left\{ \lim_{\alpha_1 \searrow 0, \ldots, \alpha_n \searrow 0} A(T_n, \alpha_n) \cdots A(T_1, \alpha_1)(f - f_k) > 4t \right\} \\ &\leqslant \mu\{(f - f_k)^* > 4t\} \\ &\leqslant C_{n-1} \int_{|f - f_k| > t} \left| \frac{f - f_k}{t} \right| \left(\log \left| \frac{f - f_k}{t} \right| \right)^{n-1} d\mu. \end{aligned}$$

Likewise

$$\mu\left\{ (f_k - \tilde{f}) > 4t \right\} \leqslant C_{n-1} \int_{|f - f_k| > t} \left| \frac{f - f_k}{t} \right| \left(\log \left| \frac{f - f_k}{t} \right| \right)^{n-1} d\mu.$$

These two inequalities imply $f_k \to \tilde{f}$ in μ-measure. By Corollary III.6.13 in [2], there exists a subsequence $\{f_{n_k}\}$ which converges to $\tilde{f}$ μ-a.e. Since $f_k \to f$ pointwise, we must have $\tilde{f}(x) = f(x)$ μ-a.e. □

Since $L_1(\mu)$ is strictly contained in R_0, the preceding theorem for the case $n = 1$ yields a slight generalization of Ornstein's local ergodic theorem [8].

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SPERRY SYSTEMS MANAGEMENT, GREAT NECK, NEW YORK 11020