PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 79, Number 2, June 1980

FREDHOLM COMPOSITION OPERATORS

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ABSTRACT. In this paper a necessary and sufficient condition for a composition operator C_T on $L^2[0, 1]$ to be a Fredholm operator is given. In addition, all Fredholm composition operators on $l^2(N)$ are characterized.

1. Preliminaries. Let $(X, \mathbb{S}, \lambda)$ be a σ-finite measure space and T be a measurable nonsingular ($\lambda T^{-1}(E) = 0$ whenever $\lambda(E) = 0$) transformation from X into itself. Then a composition transformation C_T on $L^2(X, \mathbb{S}, \lambda)$ is defined as

$$C_T f = f \circ T \quad \text{for every } f \in L^2(X, \mathbb{S}, \lambda).$$

In case C_T is a bounded operator with range in $L^2(X, \mathbb{S}, \lambda)$, we call it a composition operator induced by T. The main purpose of this paper is to study Fredholm composition operators on $L^2(X, \mathbb{S}, \lambda)$ (briefly written as $L^2(\lambda)$), where X is the unit interval, $\mathbb{S}$ is the σ-algebra of all Borel subsets of X, and λ is the Lebesgue measure on $\mathbb{S}$. A criterion for a composition operator to be Fredholm on $l^2(N)$ is also given here.

Let $B(L^2(\lambda))$, $R(C_T)^{\perp}$ and $[x, y, z, \ldots]$ denote the Banach algebra of all bounded linear operators on $L^2(\lambda)$, the orthogonal complement of the range of C_T and the closed linear span of the vectors $x, y, z, \ldots$ respectively.

DEFINITION. An operator A on a Hilbert space H is called a Fredholm operator if the range of A is closed and if the dimensions of the kernel and the cokernel are finite.

2. Fredholm composition opertors. A characterization of Fredholm composition operators on $H^2(D)$ is given by J. Cima, J. Thomson and W. Wogen in **[2]** where they proved that a composition operator C_T is Fredholm if and only if T is a conformal automorphism of the disc. The following theorem gives an analogous characterization of Fredholm composition operators on $L^2(\lambda) = L^2[0, 1]$.

THEOREM 1. *Let $C_T \in B(L^2(\lambda))$. Then C_T is a Fredholm operator if and only if it is invertible.*

PROOF. If C_T is invertible, then clearly C_T is a Fredholm operator.

It is known from [**5**, p. 82] that $C_T^* C_T = M_{f_0}$, where M_{f_0} is the multiplication operator induced by $f_0 = d\lambda T^{-1}/d\lambda$. Since X is nonatomic [**3**, pp. 171–174] and $\operatorname{Ker} C_T = \operatorname{Ker} C_T^* C_T = \operatorname{Ker} M_{f_0} = L^2(X_0)$, where $X_0 = \{x: f_0(x) = 0\}$, it follows that the dimension of the kernel of C_T of either zero or infinite.

Received by the editors May 3, 1977 and, in revised form, June 18, 1979.

AMS (MOS) subject classifications (1970). Primary 47B30; Secondary 47B30.

Key words and phrases. Composition operators, Fredholm operators, nonatomic measure, probability measure.

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0002-9939/80/0000-0266/$02.00

The map T induces an expectation operator E: $L^1(\lambda) \to L^1(\lambda T^{-1})$ defined by

$$\int (\phi \circ T) f \, d\lambda = \int \phi E(f) \, d\lambda T^{-1}$$

for $\phi \in L^\infty(\lambda T^{-1})$ [**1**, p. 847]. Since X is the unit interval, by Theorem 2 [**1**] there is a disintegration of λ with respect to T that assigns to each x in the essential range of T a probability measure λ_x on $\mathbb{S}$ such that

$$E(f)(x) = \int f \, d\lambda_x \qquad \lambda T^{-1}\text{-a.e.}$$

We now suppose C_T is a Fredholm operator. Then in order to show that C_T is invertible, it is enough to show that C_T has dense range.

Suppose $f \in L^2(\lambda)$, $f \neq 0$ and f is orthogonal to ran $C_T = L^2(X, T^{-1}(\mathbb{S}), \lambda)$, where $T^{-1}(\mathbb{S}) = \{T^{-1}(F)\colon F \in \mathbb{S}\}$ [**10**, Lemma 2.4]. Then $E(f)(x) = 0$, λT^{-1}-a.e., and

$$0 \neq \int |f|^2 \, d\lambda = \int \int |f|^2 \, d\lambda_x \, d\lambda T^{-1}(x).$$

Thus for some $\varepsilon > 0$ there is a set F of positive λT^{-1}-measure such that for $x \in F$, $\int |f|^2 \, d\lambda_x > \varepsilon$. Since $\lambda T^{-1} \ll \lambda$ and λ is nonatomic, one can write F as a countable union of pairwise disjoint sets F_n of positive λT^{-1}-measure. Since

$$\int |(\chi_{F_n} \circ T) f|^2 \, d\lambda = \int \chi_{F_n} \int |f|^2 \, d\lambda_x \, d\lambda T^{-1}(x) > \varepsilon \lambda T^{-1}(F_n) \neq 0,$$

and for every g in the range of C_T,

$$\langle (\chi_{F_n} \circ T) f, g \rangle = \int f \cdot \overline{(\chi_{F_n} \circ T) g} \, d\lambda = 0,$$

it follows that the family $\{(\chi_{F_n} \circ T) f\colon n \in N\}$ of nonzero, pairwise orthogonal functions is orthogonal to the range of C_T. Thus, either the range of C_T is dense or its orthogonal complement is infinite dimensional. This completes the proof of the theorem.

COROLLARY. *Let $C_T \in B(L^2(\lambda))$. Then C_T is a Fredholm operator if and only if T is invertible and T^{-1} induces a composition operator.*

PROOF. Proof follows from Theorem 2 of [**7**].

Before proceeding to the characterization, we shall give some examples of Fredholm composition operators on $l^2(N)$.

EXAMPLE 1. Let N be the set of natural numbers and λ be the counting measure on N. Let $l^2(N)$ denote the Hilbert space of all square summable sequences of complex numbers. Then $\{e_n\}$ is an orthonormal basis of $l^2(N)$, where

$$e_n(m) = \begin{cases} 1 & \text{whenever } m = n, \\ 0 & \text{whenever } m \neq n. \end{cases}$$

Now define a measurable transformation T as follows:

$$T(n) = \begin{cases} 1 & \text{if } n = 1, 2, 3, \\ 3 & \text{if } n = 4, 5, \\ n - 2 & \text{if } n > 6. \end{cases}$$

Now consider composition operator C_T induced by the T defined above. Ker $C_T = [e_2]$ and Ker $C_T^* = [2e_1 - e_2 - e_3, e_2 - e_3, e_4 - e_5]$. From the corollary to Theorem 2.2 **[9]** it is known that every composition operator on $l^2(N)$ has closed range. Hence C_T is a Fredholm operator.

EXAMPLE 2. Let U^* be the adjoint of the unilateral shift on $l^2(N)$. Then U^* is a composition operator induced by $T(n) = n + 1$. Ker $U^* = [e_1]$ and Ker $U = [0]$. Hence U^* is a Fredholm operator.

Now we shall give an example of a composition operator which is not a Fredholm operator.

EXAMPLE 3. Let

$$T(n) = \begin{cases} n & \text{if } n \text{ is odd,} \\ n - 1 & \text{if } n \text{ is even.} \end{cases}$$

Then Ker $C_T = [e_2, e_4, e_6, \ldots]$ and Ker $C_T^* = R(C_T)^\perp = [e_1 - e_2, e_3 - e_4, e_5 - e_6, \ldots]$. Hence C_T is a not a Fredholm operator on $l^2(N)$.

Now we shall prove the following theorem.

THEOREM 2. *Let $C_T \in B(l^2(N))$. Then C_T is a Fredholm operator if and only if the range of T contains all but finitely many elements of N and the restriction of T to the complement of some finite set is injective.*

PROOF. It follows from the corollary to Theorem 2.2 **[9]** that every composition operator on $l^2(N)$ has closed range. Since Ker C_T = Ker $C_T^* C_T$ = Ker M_{f_0} **[2]**, where $f_0(n) = \lambda T^{-1}(\{n\})/\lambda(\{n\}) = \lambda T^{-1}(\{n\})$ for $n \in N$, therefore the Ker C_T is finite dimensional if and only if $f_0(n) = 0$ for at most finitely many elements n in N. Now we claim that $\dim R(C_T)^\perp = \Sigma_m(\alpha_m - 1)$, where $m \in N_1 = \{n: \lambda T^{-1}(\{n\}) > 1\}$ and $\alpha_m = \lambda T^{-1}(\{m\})$. We give the outline of half of the proof as follows. Let $T(n_i) = m$ for $i = 1, 2, 3, \ldots, p$. Then $m \in N_1$, $\alpha_m = p$ and the characteristic function $X_{T^{-1}(\{m\})} = e_{n_1} + e_{n_2} + \cdots + e_{n_p}$ belongs to the range of C_T. It is clear that there exist $p - 1$ orthogonal vectors $f_1, f_2, \ldots, f_{p-1}$ in $[e_{n_1}, e_{n_2}, \ldots, e_{n_p}]$ which are orthogonal to the vector $X_{T^{-1}(\{m\})}$. Since $\langle f_i, X_{T^{-1}(\{n\})}\rangle = 0$ for all $n \in N$ and for $i = 1, 2, \ldots, p - 1$, it follows that the vectors $f_1, f_2, \ldots, f_{p-1}$ belong to the orthogonal complement of the range of C_T. This completes the proof of the theorem.

ACKNOWLEDGEMENTS. The author is indebted to Dr. R. K. Singh for his kind guidance during the preparation of this paper. The author also is greatly thankful to the referee of the earlier version of this paper for suggestions which contributed in an essential way to the proof of Theorem 1 presented here.

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