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ON SEPARABLE BANACH SPACES, UNIVERSAL FOR ALL SEPARABLE REFLEXIVE SPACES

J. BOURGAIN

ABSTRACT. It is shown that a separable Banach space which is universal for all separable reflexive spaces is also universal for all separable spaces.

Introduction. Our basic tool will be some results of descriptive set theory in Polish spaces. They were already used in [4]. We repeat them here, in order to make the text self-contained. Let us denote by **N** the set of positive integers.

We start with an arbitrary set X. The set $\bigcup_{n=1}^{\infty} X^n$ can be partially ordered in a natural way, by taking $(x_1, \ldots, x_n) < (x'_1, \ldots, x'_p)$ provided $p > n$ and $x_k = x'_k$ for $k = 1, \ldots, n$. Comparability and incomparability will always be related to this order.

A tree T on X will be a subset of $\bigcup_{n=1}^{\infty} X^n$ with the property that a predecessor of a member of T belongs also to T. Thus $(x_1, \ldots, x_n) \in T$ whenever $(x_1, \ldots, x_n, x_{n+1}) \in T$.

A tree T on X is well-founded provided there is no sequence $(x_n)_n$ in X satisfying $(x_1, \ldots, x_n) \in T$ for each $n \in \mathbf{N}$.

Let T be a well-founded tree on X. Proceeding by induction we associate to each ordinal α a new tree T^α: Take $T^0 = T$. If T^α is obtained, let

$$T^{\alpha+1} = \bigcup_{n=1}^{\infty} \{(x_1, \ldots, x_n) \in X^n; (x_1, \ldots, x_n, x) \in T^\alpha \text{ for some } x \in X\}.$$

If γ is a limit ordinal, define $T^\gamma = \bigcap_{\alpha<\gamma} T^\alpha$. Remark that the T^α are strictly decreasing. Hence T^α will be empty if α is sufficiently large. The ordinal $o[T]$ of T will be the smallest ordinal for which $T^{o[T]} = \emptyset$.

PROPOSITION 1. *Let T be a well-founded tree on X and take*

$$T_x = \bigcup_{n=1}^{\infty} \{(x_1, \ldots, x_n) \in X^n; (x, x_1, \ldots, x_n) \in T\} \quad \textit{for all } x \in X.$$

Then

1. $T_x^\alpha = (T^\alpha)_x$ *for each ordinal* α,
2. $o[T] = \sup_{x \in X}(o[T_x] + 1)$.

PROOF. The first statement is easily verified by induction on α. If $x \in X$ is fixed and $\alpha < o[T_x]$, then $T_x^\alpha = (T^\alpha)_x$ is nonempty and therefore $x \in T^{\alpha+1}$. Distinguishing the cases $o[T_x]$ is not a limit ordinal, $o[T_x]$ is a limit ordinal, we see that

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$x \in T^{o[T_x]}$ and hence $o[T] \geqslant o[T_x] + 1$. This proves that

$$o[T] \geqslant \sup_{x \in X} (o[T_x] + 1).$$

Let conversely $\alpha = \sup_{x \in X}(o[T_x] + 1)$. For $x \in X$, we have that $T_x^{o[T_x]} = (T^{o[T_x]})_x = \emptyset$. This means that no complexes in $T^\alpha \subset T^{o[T_x]+1}$ start with x. Thus $T^\alpha = \emptyset$ and $o[T] \leqslant \alpha$.

A tree T on a topological space X is said to be closed provided for all $n \in \mathbf{N}$ the set $T_{(n)} = \{(x_1, \ldots, x_n) \in X^n; (x_1, \ldots, x_n) \in T\}$ is closed in X^n equipped with the product topology.

Lemma 2. *Let T be a closed tree on a complete metrizable space X and assume that $T_{(n)} = \overline{\pi_n(T_{(n+1)})}$ for each $n \in \mathbf{N}$, where $\pi_n: X^{n+1} \to X^n$ denotes the projection on the first n coordinates and $^-$ the closure operation. Then either $T = \emptyset$ or T is not well-founded.*

Proof. Let d be a complete metric for X. Using the hypothesis on the sets $T_{(n)}$ it is straightforward to find for each $n \in \mathbf{N}$ some $(x_1^n, \ldots, x_n^n)$ in $T_{(n)}$ so that the following holds: $d(x_k^n, x_k^{n+1}) < 2^{-n}$ for all $k = 1, \ldots, n$ and $n \in \mathbf{N}$. Hence, for each $k \in \mathbf{N}$, the sequence $(x_k^n)_{n \geqslant k}$ converges to some point x_k in X. If we consider the sequence $(x_k)_k$ in X, then for each $k \in \mathbf{N}$ we have that $(x_1, \ldots, x_k) = \lim_{n \to \infty}(x_1^n, \ldots, x_k^n)$ and hence belongs to $T_{(k)}$ since $T_{(k)}$ is closed. Thus T is not well-founded.

We now prove a particular version of the Kunen-Martin boundedness theorem (see [6] or [3]).

Proposition 3. *If T is a well-founded closed tree on a Polish* (*separable complete metrizable*) *space, then $o[T] < \omega_1$.*

Proof. It is clear that $T_{(n)}^{\alpha+1} = \overline{\pi_n(T_{(n+1)}^\alpha)}$, where $T_{(n)}^\alpha = T^\alpha \cap X^n$. By the separability, there must be some $\eta < \omega_1$ such that $\overline{T_{(n)}^\eta} = \overline{T_{(n)}^{\eta+1}}$ for each $n \in \mathbf{N}$. Obviously $T' = \bigcup_{n=1}^\infty \overline{T_{(n)}^\eta}$ is a closed tree on X and $T'_{(n)} = \overline{T_{(n)}^\eta} = \overline{\pi_n(T'_{(n+1)})}$ for all $n \in \mathbf{N}$. Since T is closed, $T' \subset T$ and thus T' is well-founded. Therefore, by Lemma 2, $T' = T^\eta = \emptyset$.

This completes the proof.

Corollary 4. *If T is a well-founded tree on* $\mathbf{N}$, *then $o[T] < \omega_1$.*

We agree to let $o[T] = \omega_1$ if T is not well-founded.

Let us point out that for any $\alpha < \omega_1$ we can find a well-founded tree T on $\mathbf{N}$ such that $o[T] = \alpha$. The construction of the trees follows readily from the definition of the ordinal of a tree and Proposition 1.

Suppose now S a tree on a set X and T a tree on a set Y. We say that a mapping $\rho: S \to T$ is regular, provided ρ preserves the length and order of the complexes, i.e.

1. $\rho(S_{(n)}) \subset T_{(n)}$ for all $n \in \mathbf{N}$.
2. If $\rho(x_1, \ldots, x_n, x_{n+1}) = (y_1, \ldots, y_n, y_{n+1})$, then $\rho(x_1, \ldots, x_n) = (y_1, \ldots, y_n)$.

PROPOSITION 5. *If S and T are well-founded and if there exists a regular map ρ: $S \to T$, then $o[S] \leqslant o[T]$.*

PROOF. By induction, we obtain that $\rho(S^\alpha) \subset T^\alpha$ for each ordinal α. This proves our statement.

Representation of a Banach space in a given Banach space. In this section X, $\| \ \|$ is a fixed Banach space. We say that a sequence (y_n) is total in a space Y, provided $Y = \overline{\text{span}}(y_n; n)$.

We generalize the notion of l^1-tree which was introduced in [4]. Let $(Y, \| \ \|)$ be a Banach space, and let (y_n) be a sequence in Y and $\varepsilon > 0$. Consider the set $T = T(X, Y, (y_n), \varepsilon)$ of all finite complexes $(x_1, \ldots, x_n)$ of elements of X for which the following condition is fulfilled:

$$\varepsilon \left\| \sum_{k=1}^{n} a_k y_k \right\| \leqslant \left\| \sum_{k=1}^{n} a_k x_k \right\| \leqslant \varepsilon^{-1} \left\| \sum_{k=1}^{n} a_k y_k \right\|$$

for all scalars $a_1, \ldots, a_n$.

It is clear that T is a tree which is moreover closed. We let $o[X, Y, (y_n), \varepsilon]$ be the ordinal $o[T]$ of this tree T. Remark that $o[X, Y, (y_n), \varepsilon] \leqslant o[X, Y, (y_n), \varepsilon']$ if $\varepsilon' < \varepsilon$. The next property is obvious.

PROPOSITION 6. *If Y, $\| \ \|$ is a separable Banach space, then the following are equivalent*:

1. *The space X contains an isomorphic copy of Y.*

2. *There is some $\varepsilon > 0$ such that $T(X, Y, (y_n), \varepsilon)$ is not well-founded whenever (y_n) is a sequence in Y.*

3. *There is some $\varepsilon > 0$ and some total sequence (y_n) in Y such that $T(X, Y, (y_n), \varepsilon)$ is not well-founded.*

Applying Proposition 3, we find immediately:

PROPOSITION 7. *If moreover X is separable, then* (1), (2), (3) *of Proposition* 6 *are also equivalent to*

1. *There is some $\varepsilon > 0$ such that $o[X, Y, (y_n), \varepsilon]$ is not countable whenever (y_n) is a sequence in Y.*

2. *There is some $\varepsilon > 0$ and some total sequence (y_n) in Y such that $o[X, Y, (y_n), \varepsilon]$ is not countable.*

We need the following result for our purpose.

COROLLARY 8. *Assume X is a separable Banach space. Let Y be a separable Banach space, and let (y_n) be a total sequence in Y and $\varepsilon > 0$. Suppose that for all $\alpha < \omega_1$ there exists a Banach space Z_α that imbeds isomorphically in X and such that $o[Z, Y, (y_n), \varepsilon] \geqslant \alpha$. Then X contains an isomorphic copy of Y.*

PROOF. It is easily verified that if j_α: $Z_\alpha \to X$ is an isomorphic imbedding, then $o[X, Y, (y_n), \varepsilon_\alpha] \geqslant \alpha$, where $\varepsilon_\alpha = \min(\|j_\alpha\|^{-1}, \|j_\alpha^{-1}\|^{-1}) \cdot \varepsilon$. Using a standard argument, we find some $\varepsilon' > 0$ such that $\Omega = \{\alpha < \omega_1; \varepsilon_\alpha \geqslant \varepsilon'\}$ is uncountable. Thus

$o[X, Y, (y_n), \varepsilon'] \geqslant \sup_{\alpha \in \Omega} o[X, Y, (y_n), \varepsilon_\alpha]$ is uncountable and Proposition 7 completes the proof.

A result on universal spaces. We recall that a Banach space X is universal for a class $\mathcal{C}$ of Banach spaces provided each member of $\mathcal{C}$ is isomorphic to a closed subspace of X (the isomorphism constant may depend on the space in $\mathcal{C}$). In **[2]**, it is shown that $C[0, 1]$ (the space of the continuous functions on $[0, 1]$) contains every separable Banach space isometrically. W. Szlenk **[12]** proved the nonexistence of a separable reflexive space which is universal for the class $\mathcal{S}\mathcal{R}$ of the separable reflexive spaces. The precise version of his result is as follows:

THEOREM 9. *If X is universal for $\mathcal{S}\mathcal{R}$, then X^* is nonseparable.*

The basic ingredient of his proof is the notion of the "Szlenk index", which in fact turned out to have other interesting ulterior applications (cf. **[1]** and **[5]**).

We will obtain here the following improvement.

THEOREM 10. *If X is separable and universal for $\mathcal{S}\mathcal{R}$ then X is also universal for the class $\mathcal{S}$ of all Banach spaces.*

Since $C[0, 1]$ has a basis, it is sufficient to show that X contains any space with a basis isomorphically. In virtue of Corollary 8, we only have to prove the following:

PROPOSITION 11. *If Y is a Banach space and (y_n) a basis for Y with basis constant $M > 0$, then there exists for any $\alpha < \omega_1$ a separable reflexive space Z_α such that $o[Z_\alpha, Y, (y_n), (2M)^{-1}] \geqslant \alpha$.*

Thus let Y be a fixed Banach space with a fixed normalized basis (y_n).

Consider any well-founded tree S on $\mathbf{N}$. We introduce the set $\hat{S} = \bigcup_{n=1}^{\infty} \{(c_1, \ldots, c_n, c_1', \ldots, c_n');\ c_1, \ldots, c_n, c_1', \ldots, c_n' \in S$, the complexes $c_1, \ldots, c_n$ are mutually incomparable and $c_k < c_k'$ for each $k = 1, \ldots, n\}$. For $i = 0, 1, 2, \ldots$ we introduce the Banach space $Z[S; i]$ obtained by completion of the linear space of the finitely supported systems $z = (a_c)_{c \in S}$ of scalars under the norm

$$\|z\|_{S,i} = \sup \left\{ \sum_{k=1}^{n} \left\| \sum_{c_k < d < c_k'} a_d y_{|d|+i} \right\|^2 \right\}^{1/2},$$

where $|d|$ is the length of d and the sup is taken over all members $(c_1, \ldots, c_n, c_1', \ldots, c_n')$ in $\hat{S}$ (cf. **[9]** and **[10]**). For each $c \in S$ we denote by e_c the c-unit vector of $Z[S, i]$.

LEMMA 12. *The Banach spaces $Z[S; i]$ are reflexive.*

PROOF. Define $M = \{n \in \mathbf{N}; n \in S \text{ and } S_n = \varnothing\}$ and $N = \{n \in \mathbf{N}; S_n \neq \varnothing\}$ using the notations of Proposition 1. We show that $Z[S; i]$ is isomorphic to the l^2-sum of the spaces $l^2(M)$ and $(Z[S_n; i+1] \oplus \mathbf{R})_{n \in N}$. Induction on $o[S]$ will then clearly complete the proof. Assume $z = (a_c)_{c \in S}$ finitely supported and take $z_n = (a_{n,c})_{c \in S_n}$ for each $n \in N$. It is easy to verify that

$$\|z\|_{S,i} = \left\{ \sum_{n \in M} |a_n|^2 + \sum_{n \in N} \max\left[\left(\|z_n\|_{S_n,i+1} \right)^2, \sup_{c \in S, n < c} \left\| \sum_{d < c} a_d y_{|d|+i} \right\|^2 \right] \right\}^{1/2}.$$

Since now

$$|a_n| \leq \sup_{c \in S, n < c} \left\| \sum_{d < c} a_d y_{|d|+i} \right\| \leq |a_n| + \sup_{c \in S_n} \left\| \sum_{d < c} a_{n,d} y_{|d|+i+1} \right\|$$
$$\leq |a_n| + \|z_n\|_{S_n,i+1} \quad \text{for } n \in N,$$

we get

$$\left\{ \sum_{n \in M} |a_n|^2 + \sum_{n \in N} \max\left[\left(\|z_n\|_{S_n,i+1} \right)^2, |a_n|^2 \right] \right\}^{1/2}$$
$$\leq \|z\|_{S,i} \leq 2 \left\{ \sum_{n \in M} |a_n|^2 + \sum_{n \in N} \max\left[\left(\|z_n\|_{S_n,i+1} \right)^2, |a_n|^2 \right] \right\}^{1/2}.$$

Therefore the mapping $z \to ((a_n)_{n \in M}, (z_n, a_n)_{n \in N})$ gives us the required isomorphism.

LEMMA 13. $o[Z[S; 0], Y, (y_n), (2M)^{-1}] \geq o[S]$.

PROOF. It is clear that if $c \in S$ and $(c_1, \ldots, c_n, c'_1, \ldots, c'_n) \in \hat{S}$, then $c_k < c$ for at most one $k = 1, \ldots, n$. Using this fact, we see that for $c \in S$ and scalars $(a_d)_{d<c}$,

$$\left\| \sum_{d < c} a_d e_d \right\|_{S,0} = \sup_{c' < c'' < c} \left\| \sum_{c' < d < c''} a_d y_{|d|} \right\|$$

and consequently

$$\left\| \sum_{d < c} a_d y_{|d|} \right\| \leq \left\| \sum_{d < c} a_d e_d \right\|_{S,0} \leq 2M \left\| \sum_{d < c} a_d y_{|d|} \right\|.$$

But this proves that if $c \in S$, then $(e_d)_{d<c}$ is a member of $T = T(Z[S; 0], Y, (y_n), (2M)^{-1})$. Thus the map $\rho: S \to T$, $\rho(c) = (e_d)_{d<c}$ is regular and applications of Proposition 5 yield that $o[S] \leq o[T]$. This proves the lemma.

Now Proposition 11 is an immediate corollary of the two previous lemmas. This completes the proof of Theorem 10.

REMARK. We say that a Banach space X is super reflexive provided every Banach space Y which is finite dimensionally representable (f.d.r.) in X is reflexive (cf. [**8**]). P. Enflo [**7**] proved that X is super reflexive iff X admits an equivalent uniformly convex norm. It is not difficult to see that if X contains an isomorphic copy of $l^p(\mathbf{N})$ for each $p > 1$, then l^1 is f.d.r. in X (cf. [**11**, p. 91]). Consequently, there is no super reflexive space which is universal for the class $\mathbb{S}\,\mathbb{S}\,\mathfrak{R}$ of all separable super reflexive spaces. The following question seems however unsolved.

PROBLEM. Does $\mathbb{S}\,\mathbb{S}\,\mathfrak{R}$ admit a universal separable reflexive space?

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AANGESTELD NAVORSERS, N. F. W. O., VRIJE UNIVERSITEIT, BRUSSELS, BELGIUM