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THE LENGTH OF A CURVE IN A SPACE OF CURVATURE $\leqslant K$

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ABSTRACT. Let M be a compact ball in a Riemannian manifold with sectional curvatures $\leqslant K$. Suppose its radius R_0 is less than the injectivity radius at the center of M and $R_0 < \pi/2\sqrt{K}$ if $K > 0$. Denote by M_0 a circle of radius R_0 in the plane of constant curvature K and by κ the curvature of ∂M_0. Then any curve in M with curvature $\leqslant \chi < \kappa$ is not longer than a circular arc of curvature χ in M_0 whose ends are opposite points of ∂M_0. Any curve in M with total curvature not exceeding some $\tau > 0$ ($\tau = \pi/2$ if $K \leqslant \kappa^2$) is not longer than the longest curve in M_0 with the same total curvature whose tangent vector rotates in a permanent direction.

0. Introduction.

0.1. We establish here (Theorems 1.2, 1.3, 1.6) upper bounds of the length of a curve in a (convex) ball in a Riemannian space with sectional curvatures (in the ball) $\leqslant K$. The upper bounds are expressed in terms of K, the radius of the ball and curvature of the curve.

Some applications of these estimates are brought out in §2. Say any infinitely long curve in a simply connected complete space of curvature $\leqslant K \leqslant 0$ goes to ∞ if its total curvature does not grow "too fast" (or just finite), see 2.7. We establish in 2.8 how fast the curve goes to ∞ in terms of its curvature.

0.2. Results of a similar nature are proved in [2] and [3]. The difference is that there the ball is replaced by a more general convex region, instead of K a lower bound k_s of sectional curvatures in the region is used and the role of the radius is played by the minimal normal curvature κ of the boundary of the region. Moreover, the condition $k_s > -\kappa^2$ is supposed to hold in [2], [3].

We apply here a modification of the method used in [3] (rolling a curve along a fixed one, see [3, 3.11]).

0.3. As for the history of the problem, the length of a curve in a 2-dimensional surface was estimated by A. D. Aleksandrov and V. V. Strel'cov [1] in 1953. Their estimates and ours (when the dimension $n = 2$) do not follow from one another.

In 1969, Gromoll and Meyer [4, Lemma 6] proved that for any compact set in a complete open manifold of positive curvature there exists a number such that the length of any geodesic in the set is less than that number.

0.4. In Euclidean case, a similar result is represented by Rešetnyak Theorem [5, p. 262]. Its simplified version is as follows.

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REŠETNYAK THEOREM. *Let x: $[0, L] \to R^n$ be a piecewise regular curve parametrized by its arc length. Put $\delta = \max \sphericalangle(\dot{x}(a), \dot{x}(b))$ where $\sphericalangle$ means angle and the maximum is taken over all regularity points $a, b \in [0, L]$. If all vectors $\dot{x}(s)$ (in regularity points) are directed into the same half-space and $\cos\delta > -1/(n-1)$, then*

$$L \leqslant \frac{r\sqrt{n}}{\sqrt{1 + (n-1)\cos\delta}} \tag{0.1}$$

where r is the distance between $x(0)$ and $x(L)$.

1. Curves in a ball.

1.1. *Basic notation and assumptions.* All manifolds and curves here are supposed to be of class C^∞ unless otherwise stated. A curve parametrized by arc length will be called *normal*. An oriented 2-dimensional sphere, plane or hyperbolic plane of curvature K will be denoted by P^2.

For a normal curve c: $[0, L] \to P^2$, along with the ordinary curvature $|\ddot{c}|$, we consider *oriented curvature*, i.e., $|\ddot{c}|$ with ascribed sign + (–) if $\dot{c}$ rotates in the positive (negative) direction. Total curvature of a curve and its total oriented curvature (if it lies in P^2) are integrals of the appropriate curvatures along the curve. These definitions are naturally generalized for a piecewise C^2-curve.

A minimal geodesic with the ends a, b (in any space considered) is denoted sometimes by ab, its length by $\overline{ab}$ and its direction (unit vector at a point in ab being under discussion) by $\overrightarrow{ab}$. The notations $\sphericalangle(\cdot,\cdot)$ and $\sphericalangle \ldots$ mean angle.

Throughout the paper, we denote by M a compact ball in a Riemannian manifold and set $\Gamma = \partial M$. Let K be an upper bound of sectional curvatures in M and let R_0 be its radius. We suppose R_0 to be less than injectivity radius at the center of M and

$$R_0 < \frac{\pi}{2\sqrt{K}} \quad \text{if } K > 0. \tag{1.1}$$

We assign to M a circle $M_0 \subset P^2$ of radius R_0 and denote by $\kappa = \kappa(R_0, K)$ the curvature of its circumference Γ_0. By (1.1), the circle M_0 is convex and $\kappa > 0$. It follows easily from the Rauch Comparison Theorem that M is convex as well.

The distance in M is denoted by $\rho(\cdot,\cdot)$ and in P^2 by $\rho_0(\cdot,\cdot)$.

1.2. The following three theorems coincide literally with Theorems 1.5, 1.6, 1.10 in [3] after replacement of k_s there by K (although M and M_0 here have another meaning).

THEOREM. *Any curve in M with curvature at every point not greater than $\chi \in [0, \kappa)$ is not longer than a circular arc in M_0 of curvature χ whose ends are opposite points of Γ_0. (In particular, any geodesic in M is not longer than $2R_0$.)*

1.3. THEOREM. *Let a curve of length L lie in M and have the total curvature θ satisfying*

$$\theta \in \begin{cases} [0, \theta^*) & \text{if } K > \kappa^2, \\ [0, \pi/2] & \text{otherwise,} \end{cases} \tag{1.2}$$

where the precise value of $\theta^* = \theta^*(\kappa, K)$ is given in 1.4. *Denote by l_θ supremum of the lengths of piecewise C^2-curves in M_0 such that the straight line of support of each curve rotates in the same direction when moving along the curve and the total curvature of each curve does not exceed θ. Then l_θ is finite and*

$$L \leqslant l_\theta. \tag{1.3}$$

1.4. The condition $K > \kappa^2$ $(= K \cot^2 \sqrt{K}\, R_0)$ in (1.1) implies $2R_0 > \pi/2\sqrt{K}$. Let Y_0Z_0 be a diameter of M_0 and Z_0V_0 be a chord of length $\pi/2\sqrt{K}$. Put $\delta = \sphericalangle Y_0Z_0V_0$ and denote by σ the area of the part of M_0 bounded by Y_0Z_0, Z_0V_0 and the (shortest) arc $\overset{\smile}{V_0Y_0} \subset \Gamma_0$. We set $\theta^* = \pi/2 + \delta - K\sigma$. It was shown in [3, 2.4] that $\theta^* \in (0, \pi/2)$.

1.5. Denote by J_θ the class of the curves in M_0 mentioned in 1.3. Proposition 4.2 in [3] (with k_s replaced by K) establishes that there is a longest curve γ_θ in the class J_θ. It gives also the exact description of γ_θ. The description shows that the length l_θ of γ_θ increases by θ. In the case $K \leqslant \kappa^2$, γ_θ is a polygonal line ACB with the ends A, B in diametrically opposite points of Γ_0, $\overline{AC} = \overline{CB}$ and with the total curvature $\pi - \sphericalangle ACB = \theta$.

1.6. A nondecreasing piecewise C^1-function Ξ: $[0, \infty) \to R$ with $\Xi(0) = 0$ will be called a turn-function if it is continuous from the left and any of its jumps is less than π.

Our main technical result is the following:

THEOREM. *Let $d \in [0, R_0]$, $\alpha \in [0, \pi]$ and let Ξ be a turn-function. Denote by M_0^+ the closed semicircle separated from M_0 by a diameter Y_0Z_0 and such that rotation of the radius $0Y_0$ to the radius $0Z_0$ within M_0^+ is positive. Suppose, there exists a mapping γ_0 such that*

(i) *γ_0 is a normal piecewise C^2-curve* $[0, \infty) \to P^2$,

(ii) *$\gamma_0(0) \in 0Y_0$ and $\overline{Y_0\gamma_0(0)} = d$,*

(iii) *$\dot\gamma_0$ is directed into M_0^+ and $\sphericalangle(\dot\gamma_0(0), \overrightarrow{0Y_0}) = \alpha$,*

(iv) *the total oriented curvature of $\gamma_{0|[0,s]}$ is equal to $\Xi(s)$, $s \geqslant 0$,*

(v) *there is a maximum number $L_0 > 0$ such that $\gamma_0([0, L_0]) \subset M_0^+$,*

(vi) *$\gamma_0(L_0) \in \Gamma_0 \cap M_0^+ \setminus Z_0$ and the curve $\gamma_{0|[0,L_0]}$, the segment $\gamma_0(0)Z_0$ and the arc $\gamma_0(\overset{\smile}{L_0})Z_0$ of the semicircle $\Gamma_0 \cap M_0^+$ bound a nondegenerate region. (This implies that the region is convex and $\alpha < \pi$.)*

Then the length L of any normal curve γ: $[0, L] \to M$ does not exceed L_0 if

(I) $\rho(\gamma(0), \Gamma) = d$,

(II) *in the case $d \neq R_0$, $\dot\gamma(0)$ forms an angle $\phi \leqslant \alpha$ with the radius coming through $\gamma(0)$ and directed from the center of M,*

(III) $|\ddot\gamma(t)| \leqslant \Xi'(t)$ $(= |\ddot\gamma_0(t)|)$ *for those $t \in [0, L]$ where Ξ' exists.*

2. Some remarks and applications.

2.1. Theorems 1.3 and 1.6 are naturally generalized for the case of a piecewise C^2-curve in M.

2.2. Let γ: $[0, L] \to M$ be a normal piecewise C^2-curve of total curvature $\theta > 0$. Let $\theta = \sum_{i=1}^m \theta_i$ where each θ_i satisfies (1.2). Then $L \leqslant \sum_{i=1}^m l_{\theta_i}$ where l_{θ_i} is as in Theorem 1.3.

Indeed, γ can be divided into m parts of total curvature $\tau_i \leqslant \theta_i$, $i = 1, 2, \ldots, m$. Applying Theorem 1.3 to each part and adding, we have

$$L \leqslant \sum_{i=1}^{m} l_{\tau_i} \leqslant \sum_{i=1}^{m} l_{\theta_i},$$

see 1.5.

2.3. The equalities hold in the estimates of 1.2, 1.3 and 1.6 when $M = M_0$ and the curve in M mentioned there coincides with a circular arc of curvature χ having the ends in opposite points of Γ_0, with γ_θ (see 1.5) and $\gamma_{0|[0,L_0]}$, respectively.

2.4. Let for example M be a ball of radius R_0 in R^n and $\theta = \pi/2$. Take $K = 0$. Then M_0 is a circle of radius R_0 in R^2 and we may imbed M_0 into M. According to 1.5, $l_\theta = l_{\pi/2} = 2\sqrt{2}\, R_0$. So, by Theorem 1.3, any curve with total curvature $\leqslant \pi/2$ in the ball M has the length $\leqslant 2\sqrt{2}\, R_0$. This estimate is realized by a polygonal line $ACB \subset M_0 \subset M$ with $A, C, B \subset \partial M_0 \subset \partial M$, $\overline{AC} = \overline{CB}$ and $\sphericalangle ACB = \pi/2$.

The Rešetnyak Theorem 0.4 (which is also exact) applied to ACB yields a rougher estimate:

$$\overline{AC} + \overline{CB} \leqslant \frac{2R_0\sqrt{n}}{\sqrt{1 + (n-1)\cos \pi/2}} = 2\sqrt{n}\, R_0.$$

2.5. In the rest of §2, N means a complete simply connected Riemannian space with sectional curvatures $\leqslant K \leqslant 0$. Let $\gamma\colon [0, \infty) \to N$ be a normal curve, $\xi(s)$ be its curvature and $\theta(s) = \int_0^s \xi(x)dx$. We denote by $B_t(p)$ the closed ball of radius t centered at $p \in N$. By the Hadamard-Cartan Theorem, it is homeomorphic to a ball.

2.6. Let us show that γ goes to ∞ if $\overline{\lim}_{s\to\infty} \xi(s) \leqslant \sqrt{-K}$. Notice that $\kappa(R_0, K)$ (see 1.1) decreases by R_0 and $\lim_{R_0\to\infty} \kappa(R_0, K) = \sqrt{-K}$. Suppose now the contrary, i.e., that γ can be included in a ball $B_t(p)$. The curvature $\kappa(t, K)$ of the boundary of the circle M_0 (constructed for the ball $M = B_t(p)$) is greater than $\sqrt{-K}$. Then $\xi(s) < \kappa(t, K)$ for s larger than some s_0. By Theorem 1.2, any piece of $\gamma_{|[s_0, \infty)}$ is not longer than the arc mentioned there which is impossible. (The remark 2.6 can be proved without reference to Theorem 1.2 and probably is known.)

2.7. In 2.7–2.8, we take $K = 0$. Let us represent $\theta(s)$ as $(m-1)(\pi/2) + \tilde{\theta}$ where m is an integer and $\tilde{\theta} \in [0, \pi/2)$. If $\gamma_{|[0,s]}$ lies in a ball $B_t(p)$ then, by 2.2,

$$s \leqslant (m-1)l_{\pi/2} + l_{\tilde{\theta}} < m l_{\pi/2} \leqslant \left(\frac{\theta(s)}{\pi/2} + 1\right) \cdot 2\sqrt{2}\, t. \tag{2.1}$$

We can see now that γ goes to ∞ if $\underline{\lim}_{s\to\infty} \theta(s)/s = 0$. Indeed, otherwise $\gamma([0, \infty)) \subset B_t(p)$ and (2.1) holds for any $s > 0$ so that

$$\underline{\lim}_{s\to\infty} \frac{\theta(s)}{s} \geqslant \underline{\lim}_{s\to\infty} \left(\frac{1}{2\sqrt{2}\, t} - \frac{1}{s}\right)\frac{\pi}{2} = \frac{\pi}{4\sqrt{2}\, t} > 0.$$

2.8. Suppose there are numbers $A > 0$, $\alpha \in [0, 1)$ such that $\theta(s) \leqslant As^\alpha$ for sufficiently large s. By 2.7, γ goes to ∞. Take an arbitrary point $p \in N$. For $t > \overline{p\gamma(0)}$, denote by $s(t)$ the maximum number such that $\gamma([0, s(t)]) \subset B_t(p)$. Then

$$\overline{\lim_{t\to\infty}}\ \frac{s(t)}{t^{1/(1-\alpha)}} \leqslant \begin{cases} \left(\dfrac{2\sqrt{2}\,A}{\pi/2}\right)^{1/(1-\alpha)} & \text{if } \alpha \in (0, 1), \\ 2\sqrt{2}\left(\dfrac{A}{\pi/2} + 1\right) & \text{if } \alpha = 0. \end{cases} \tag{2.2}$$

PROOF. By (2.1),

$$s(t) \leqslant \left(\frac{As^{\alpha}(t)}{\pi/2} + 1\right)2\sqrt{2}\ t. \tag{2.3}$$

Suppose first $\alpha \neq 0$. Fix some $\varepsilon > 0$. When t (and $s(t)$) are sufficiently large then $(As^{\alpha}(t)/(\pi/2))\varepsilon > 1$. Now (2.3) implies

$$s(t) \leqslant \frac{As^{\alpha}(t)}{\pi/2}(1 + \varepsilon)2\sqrt{2}\ t,$$

$$s(t) \leqslant \left[(1 + \varepsilon)\frac{2\sqrt{2}\,A}{\pi/2}\,t\right]^{1/(1-\alpha)}.$$

Dividing this by $t^{1/(1-\alpha)}$, passing to the upper limit as $t \to \infty$ and then to the limit as $\varepsilon \to 0$, one gets (2.2) for $\alpha \in (0, 1)$.

For $\alpha = 0$, (2.2) is proved by dividing (2.3) by t and passing to the upper limit.

Notice that the case $\alpha > 1$ and the case $\alpha = 1$ with sufficiently large A are not of interest since then in N there always exists a curve γ (any periodic curve) satisfying $\theta(s) \leqslant As^{\alpha}$ but not going to ∞.

3. Proofs of Theorems of §1.

3.1. *Proof of Theorem* 1.2. Let b: $[0, L] \to M$ be a normal curve with $|\ddot{b}| \leqslant \chi < \kappa$. Let $b(\lambda)$, $\lambda \in [0, L]$, be a point closest to the center C of M. Put $\gamma(t) = b(\lambda + t)$, $t \in [0, L - \lambda]$. Let the radius $CY \ni \gamma(0)$. Obviously, $\sphericalangle(\dot{\gamma}(0), \overrightarrow{CY}) = \pi/2$. (If $\gamma(0) = C$, we choose a radius with this property.) Since $\chi < \kappa$, there exists the circumference γ_0 of curvature χ satisfying the conditions (i)–(vi) of Theorem 1.6 for the set $d = \overline{Y\gamma(0)}$, $\alpha = \pi/2$, $\Xi(t) = \chi t$. Since $|\ddot{\gamma}(t)| = |\ddot{b}(\lambda + t)| \leqslant \chi = \Xi'$, the conditions (I)–(III) hold as well. Then $L - \lambda \leqslant L_0$ (see 1.6 (v)).

Similarly, $\lambda \leqslant L_0$. Addition yields $L \leqslant 2L_0$. But $2L_0$ is the length of a circular arc of curvature χ in M_0 consisting of $\gamma_0([0, L_0])$ and its specular reflection with respect to the diameter Y_0Z_0. Obviously, this arc is not longer than the one mentioned in Theorem 1.2.

3.2. *Proof of Theorem* 1.3. It coincides literally with Proof 2.5 (along with Lemma 2.2) in [3] if k_s there is replaced by K and the reference to Theorem 1.10 there is replaced by the reference to Theorem 1.6 of this paper. Moreover, since M is a ball here, some obvious simplifications are available.

3.3. *Proof of Theorem* 1.6. It is close to the proof of Theorem 1.10 in [3, §3] but much simpler and shorter since M is a ball here. A principal difference will be remarked on in §3.6.

By a simple limit reasoning, we may assume that $\Xi \in C^1$, (then $\gamma_0 \in C^2$) and that $\gamma([0, L]) \subset \operatorname{int} M \setminus C$ where C is the center of M. We assume also that there are no points where γ is tangent to the radii of M. For the dimension $n \geqslant 3$, such

points can be easily eliminated by variation of γ. For $n = 2$, a finite number of such points can be inevitable but they may be treated as in [3].

3.4. Denote by p, ω the polar coordinates in P^2 with the pole at the center 0 of M_0 and the angle ω counted in the positive direction from the radius $0Y_0$. Put $r(t) = \rho(\gamma(t), \Gamma)$. By 3.3, $r(t) \in C^\infty$, $r(t) \in (0, R_0)$ and $|r'| < 1$, $t \in [0, L]$. Consider the curve γ_1: $[0, L] \to M_0$ with the equations

$$p = R_0 - r(t), \tag{3.1}$$

$$\omega = \begin{cases} \sin\left(R_0\sqrt{K}\right)\displaystyle\int_0^t \frac{\sqrt{1 - r'^2(x)}}{\sin(R_0 - r(x))\sqrt{K}}\,dx & \text{if } K > 0, \\ R_0\displaystyle\int_0^t \frac{\sqrt{1 - r'^2(x)}}{R_0 - r(x)}\,dx & \text{if } K = 0, \\ \sinh\left(R_0\sqrt{-K}\right)\displaystyle\int_0^t \frac{\sqrt{1-r'^2(x)}}{\sinh(R_0-r(x))\sqrt{-K}}\,dx & \text{if } K < 0. \end{cases} \tag{3.2}$$

Obviously, $\gamma_1 \in C^\infty$. One can check that $|\dot\gamma_1| = 1$. So, γ_1 is a normal curve. Moreover, $\sphericalangle(\dot\gamma_1(0), \overrightarrow{0Y_0}) = \cos^{-1} - r'(0) = \phi$ and $\gamma_1([0, L]) \in \operatorname{int} M_0$.

3.5. A simple calculation based on (3.1), (3.2) shows that the oriented curvature $\xi_1(t)$ of γ_1 satisfies

$$r'' = \sqrt{1 - r'^2}\,\xi_1 - (1 - r'^2)\kappa_r \tag{3.3}$$

where κ_r (> 0) is curvature of the circumference of radius $R_0 - r$ centered at 0. (It comes through $\gamma_1(t)$.) By (3.9) in [2],

$$r'' < \sqrt{1 - r'^2}\ |\ddot\gamma| - (1 - r'^2)x \tag{3.4}$$

where x is the normal curvature of the sphere (coming through $\gamma(t)$) of radius $R_0 - r$ centered at C on the side of the interior normal ν in the direction of $\dot\gamma - \nu\langle\dot\gamma, \nu\rangle$ ($\neq 0$ since $|\langle\dot\gamma, \nu\rangle| < 1$, see 3.3).

It follows easily from Rauch Comparison Theorem that $x \geqslant \kappa_r$. Replacing x by κ_r in (3.4) and combining it with (3.3), one obtains $\xi_1 < |\ddot\gamma|$. Along with the conditions (III) and (iv) of 1.6, it yields

$$\xi_1(t) < \Xi'(t) = \xi_0(t), \qquad t \in [0, L], \tag{3.5}$$

where ξ_0 is the oriented curvature of γ_0.

3.6. The inequality (3.5) is what provides a desired property of rolling γ_0 along γ_1. It is the basis of the further geometric construction.

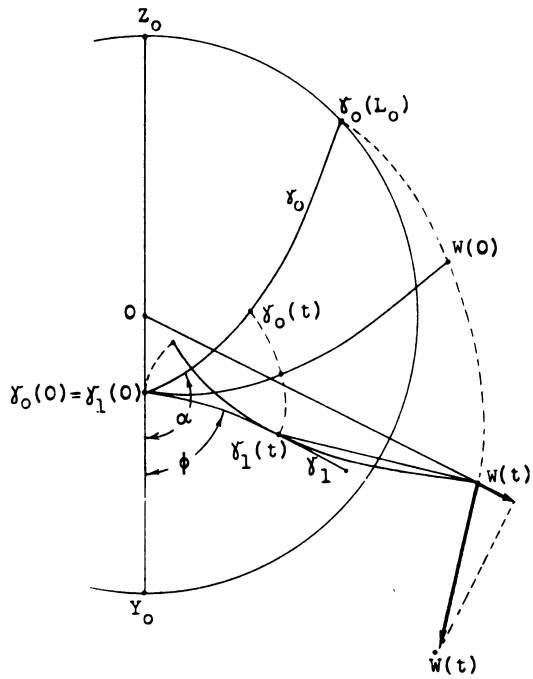
Notice that the inequality $x \geqslant \kappa_r$ from which (3.5) follows appeared in [2] and [3] as well but due to other reasons. In [2], the normal curvature x (denoted there by K) of the surface $\Gamma(r)$ parallel to Γ and distant by r from Γ satisfies $x \geqslant \kappa_r$ for the following two reasons.

(1) By the construction, the curvature of Γ_0 there is a lower bound of normal curvatures of Γ.

(2) The curvature (k_s) of P^2 there is a lower bound of sectional curvatures in M.

Then (1) and (2) result in $x \geqslant \kappa_r$ by a version of the Rauch Comparison Theorem dealing with Jacobi fields associated with a hypersurface (Γ). (See [2, (3.9), Lemma 4].)

3.7. To finish the proof, it is enough to show that the curve $\gamma_{0|[0,L_0]}$ is not longer than γ_1. We describe here only the idea of the reasoning. (See [3, §3] for details.)



Let us rotate γ_0 about its end $\gamma_0(0)$ in the negative direction about the angle $\alpha - \phi \geqslant 0$. Denote by $W(0)$ the new position of the point $\gamma_0(L_0)$, see the figure. Obviously, $W(0) \notin \operatorname{int} M_0$.

Now let γ_0 roll along γ_1 assuming that at a moment t the new position of $\gamma_0(t)$ coincides with $\gamma_1(t)$. Instantaneously, γ_0 rotates about the point $\gamma_1(t)$ in the negative direction with the angular speed $\xi_0(t) - \xi_1(t) \geqslant 0$, see (3.5). Let $W(t)$ be the position of $\gamma_0(L_0)$ at a moment t and $\dot{W}(t)$ be its speed. Then $\dot{W}(t) \perp W(t)\gamma_1(t)$ and therefore $\dot{W}(t)$ has a nonnegative projection on the vector $\overrightarrow{0W(t)}$, see the figure. It means that $W(t)$ does not get closer to the pole 0, so that $W(t) \notin \operatorname{int} M_0$.

If now $L_0 < L$ then, at the moment $t = L_0$, one has $W(L_0) = \gamma_1(L_0) \in \operatorname{int} M_0$ which contradicts what is said above.

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