PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 79, Number 2, June 1980

$\bar{P} = \bar{R}$ FOR MAPS OF THE INTERVAL

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ABSTRACT. We show that for continuous maps of a compact interval to itself the closure of the set of periodic points coincides with the closure of the set of recurrent points.

Let $f: I \to I$ be a continuous map of a compact interval to itself. Let P be the set of periodic points and R the set of recurrent points. The main result of this paper is

THEOREM 1. *For continuous maps of a compact interval to itself,* $\bar{P} = \bar{R}$.

Special cases of this theorem have been proved by L. Block **[Bl]** (for maps with finite nonwandering set), by the authors **[CH]** (for maps whose set of periodic points is closed) and by L.-S. Young **[Y]** (for piecewise monotonic maps).

As a corollary, we show that in some sense all of the interesting dynamical behavior of f occurs on $\bar{P}$.

Let NW be the set of nonwandering points and W the set of wandering points. We show (Theorem 2) that W is open and dense in $I - \bar{P}$ and hence that $NW - \bar{P}$ is nowhere dense in I. For an example in which $NW - \bar{P} \neq \varnothing$, see **[Y]**.

Recall that $P = \{x | f^n(x) = x \text{ for some } n \geq 1\}$, $R = \{x |$ for each neighborhood U of $x, f^n(x) \in U$ for some $n \geq 1\}$. Clearly $P \subseteq R$ and both sets are invariant, i.e., $f(P) \subseteq P$ and $f(R) \subseteq R$. It is well known **[ES]** that for every $n \geq 1$, $x \in R$ if and only if x is recurrent under f^n.

To prove Theorem 1 it is sufficient to show that $J \cap R = \varnothing$ for every component J of $I - \bar{P}$. For then $R \subseteq \bar{P}$ and the result follows. We shall do this with the aid of a series of lemmas.

Lemma 1 is an immediate consequence of the Intermediate Value Theorem.

LEMMA 1. *Let J be an interval such that $J \cap P = \varnothing$. Then for each $n \geq 1$, either $f^n(x) > x$ for all $x \in J$ or $f^n(x) < x$ for all $x \in J$.*

The components of $I - \bar{P}$ are intervals which are open in I. Lemmas 2–4 give conditions which are sufficient for such an interval to contain no recurrent points.

LEMMA 2. *Let J be an interval which is open in I. If for some $n \geq 1$, either $f^{kn}(x) > x$ for all $x \in J$ and all $k \geq 1$, or $f^{kn}(x) < x$ for all $x \in J$ and all $k \geq 1$, then $J \cap R = \varnothing$.*

PROOF. It suffices to show that no point of J is recurrent under f^n, for then **[ES]** no point of J is recurrent.

Received by the editors June 12, 1979.
1980 *Mathematics Subject Classification*. Primary 54H20.

0002-9939/80/0000-0283/$01.75

Let $x \in J$. If $f^{kn}(x) \notin J$ for all $k \geq 1$, then since J is open, x is not recurrent under f^n. Suppose then that $k \geq 1$ is the least positive integer such that $f^{kn}(x) \in J$. Without loss of generality, $f^{kn}(x) > x$. Then $f^{jn}(x) > f^{kn}(x) > x$ for all $j > k$. Hence if $U = J \cap (-\infty, f^{kn}(x))$, then $f^{mn}(x) \notin U$ for all $m \geq 1$. Thus x is not recurrent under f^n. □

LEMMA 3. *Let J be an interval which is open in I. If $J \cap P = \varnothing$ and $f^n(J) \subseteq J$ for some $n \geq 1$, then $J \cap R = \varnothing$.*

PROOF. Let $x \in J$. Since $J \cap P = \varnothing$, $f^n(x) \neq x$. Without loss of generality, $f^n(x) > x$ and hence by Lemma 1, $f^n(y) > y$ for all $y \in J$. Since $f^{kn}(x) \in J$ for all $k \geq 1$, $f^{kn}(x) > f^{(k-1)n}(x) > \cdots > x$. Again by Lemma 1, $f^{kn}(y) > y$ for all $y \in J$ and all $k \geq 1$. By Lemma 2, $J \cap R = \varnothing$. □

LEMMA 4. *Let J be an interval which is open in I. If $J \cap P = \varnothing$ and $f^n(J) \cap P \neq \varnothing$ for some $n \geq 1$, then $J \cap R = \varnothing$.*

PROOF. Suppose $x \in J$ and $f^n(x) \in P$. Choose $m \geq n$ so that $f^{m+n}(x) = f^n(x)$. Without loss of generality, $f^m(x) > x$. Then for all $k \geq 1$, $f^{km}(x) = f^m(x) > x$ and hence by Lemma 1, $f^{km}(y) > y$ for all $y \in J$ and all $k \geq 1$. By Lemma 2, $J \cap R = \varnothing$. □

To complete the proof of Theorem 1, let J be a component of $I - \bar{P}$. We show that $J \cap R = \varnothing$.

If $f^n(J) \cap J = \varnothing$ for all $n \geq 1$, then since J is open, $J \cap R = \varnothing$. If $f^n(J) \subseteq J$ for some $n \geq 1$, then by Lemma 3, $J \cap R = \varnothing$. Suppose then that for some $n \geq 1$, $f^n(J) \cap J \neq \varnothing$ and $f^n(J) \cap (I - J) \neq \varnothing$.

Since $f^n(J)$ is an interval, it contains an endpoint q of J, $q \notin J$. If $q \in P$, then by Lemma 4, $J \cap R = \varnothing$. Suppose then that $q \notin P$. Without loss of generality, $f^n(x) > x$ for all $x \in J$ and hence by Lemma 1 for all $x \in J' = J \cup \{q\}$. Thus q is the right-hand endpoint of J and $f^n(q) > q$. Therefore $f^n(J)$ contains points greater than q and points less than q, and hence all points sufficiently close to q. Since $q \in \bar{P}$, there are points of P arbitrarily close to q. Hence $f^n(J) \cap P \neq \varnothing$ and so by Lemma 4, $J \cap R = \varnothing$.

This completes the proof of Theorem 1.

Recall that

$$NW = \{x \mid \text{for every neighborhood } U \text{ of } x,\ f^n(U) \cap U \neq \varnothing \text{ for some } n \geq 1\},$$
$$W = I - NW.$$

Clearly $R \subseteq NW$, NW is closed and invariant and $f|\bar{R}$ is pointwise nonwandering.

Let $X_0 = I$. For α a successor ordinal, let X_α be the nonwandering set of $f|X_{\alpha-1}$. For α a limit ordinal, let $X_\alpha = \bigcap_{\beta<\alpha} X_\beta$. There exists a countable ordinal γ such that $X_\gamma = X_{\gamma+1} = \cdots$. (This construction is due to G. D. Birkhoff [**Bi**, Chapter VII] who used it to define the set of "central motions" of a dynamical system. For this reason, we call X_γ the *Birkhoff center* of f.) By [**GH**, 7.20] $X_\gamma = \bar{R}$. It follows that if E is a closed invariant subset with the property that $f|E$ is pointwise nonwandering, then $E \subseteq \bar{R}$.

It follows from the construction above and repeated applications of the fact that $\mu(f|NW) = 1$ for every invariant measure μ that $\mu(\bar{R}) = 1$ for any such measure. Then by the Sup Theorem for topological entropy **[D]**, $h(f) = h(f|\bar{R})$.

Therefore we have the following corollary to Theorem 1.

COROLLARY. *For continuous maps of a compact interval to itself,*

(1) *Every minimal set is contained in* $\bar{P}$.

(2) $f|\bar{P}$ *is pointwise nonwandering. No larger closed invariant subset has this property.*

(3) $\mu(\bar{P}) = 1$ *for every (normalized) invariant measure* μ. *No smaller closed invariant subset has this property.*

(4) $h(f) = h(f|\bar{P})$ *where* $h(\ )$ *denotes topological entropy.*

THEOREM 2. *For continuous maps of a compact interval to itself,* W *is open and dense in* $I - \bar{P}$, *and* $NW - \bar{P}$ *is nowhere dense in* I.

To prove Theorem 2, it suffices to show that W is dense in $I - \bar{P}$.

LEMMA 5. *Let* K *be an interval which is open in* I. *If* $K \subseteq NW$, *then* $[f(K)]^\circ \neq \varnothing$.

PROOF. If not, then $f(K)$ is a point, call it x. Since K is open and $K \subseteq NW$, $f^n(K) \cap K \neq \varnothing$ for some $n \geqslant 1$. Thus $f^{n-1}(x) \in K$ and hence $f^n(x) = x$. Let $U = K - \{x, \ldots, f^{n-1}(x)\}$. Then $\varnothing \neq U \subseteq NW$, U is open and $f^m(U) \cap U = \varnothing$ for all $m \geqslant 1$. This is a contradiction. □

It follows easily from Lemma 5 that $\overline{(NW)^\circ}$ is invariant. Since $f|\overline{(NW)^\circ}$ is pointwise nonwandering, $\overline{(NW)^\circ} \subseteq \bar{R} = \bar{P}$. Therefore W is dense in $I - \bar{P}$.

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