A COMBINATORIAL PROOF OF SCHUR'S 1926
PARTITION THEOREM

DAVID M. BRESSOUD

ABSTRACT. One of the partition theorems published by I. J. Schur in 1926 is an extension of the Rogers-Ramanujan identities to partitions with minimal difference three. This theorem of Schur is proved here by establishing a one-to-one correspondence between the two types of partitions counted.

Many classical partition identities state that for each positive integer n the partitions of n with parts restricted to certain residue classes are equinumerous with the partitions of n on which certain difference conditions are imposed. Most prominent among these are Euler's identity: partitions into odd parts are equinumerous with partitions into distinct parts; and the Rogers-Ramanujan identities: for $r = 1$ or 2, partitions into parts congruent to $\pm r$ (mod 5) are equinumerous with partitions into parts with minimal difference two and smallest part greater than or equal to r. In 1926, I. J. Schur discovered the partition theorem which is a natural extension [2].

SCHUR'S THEOREM. *Given any positive integer n, the partitions of n into parts congruent to ± 1 (mod 6) are equinumerous with the partitions of n into parts with minimal difference three and difference at least six between multiples of three.*

Many proofs of Euler's identity exist, including several which explicitly exhibit the correspondence between the two types of partitions counted (e.g., see [1, §19.4]). For the Rogers-Ramanujan identities, no such correspondence has been established. The purpose of this paper is to explicitly exhibit a one-to-one correspondence between the two types of partitions counted in Schur's Theorem. What makes the correspondence possible for Schur's identity and not for the Rogers-Ramanujan identities is that Schur's identity is a special case of the following partition theorem in which the parts restricted to certain residue classes are distinct.

SCHUR'S THEOREM (FULL GENERALITY). *Given positive integers r and m such that $r < m/2$, let $C_{r,m}(n)$ denote the numbers of partitions of n into distinct parts congruent to $\pm r$ (mod m), and let $D_{r,m}(n)$ denote the number of partitions of n into distinct parts congruent to 0, $\pm r$ (mod m) with minimal difference m, minimal difference $2m$ between multiples of m. Then $C_{r,m}(n) = D_{r,m}(n)$ for all n.*

The special case given above is $r = 1$, $m = 3$. Clearly, $D_{1,3}(n)$ counts the partitions of n with minimal difference three, minimal difference six between

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multiples of three. To see that $C_{1,3}(n)$ also counts the partitions of n into parts congruent to $\pm 1 \pmod 6$, the reader is referred to the correspondence proof of Euler's identity given in Hardy and Wright [1, §19.4]. Following the correspondence given there, a one-to-one correspondence between odd parts with no multiples of three (i.e. parts $\equiv \pm 1 \pmod 6$) and distinct parts with no multiples of three (i.e. distinct parts $\equiv \pm 1 \pmod 3$) is easily established.

The correspondence given below will be for Schur's Theorem in its full generality.

DEFINITION. Given positive integers r, m, n and k such that $r < m/2$, $n - mk(k - 1)/2 > rk$, an r, m *underlying partition for n and k*, or an *underlying partition*, is a partition of $n - mk(k - 1)/2$ into exactly k parts, each of which is congruent to $0, \pm r \pmod m$, and such that multiples of m are distinct. (Example: 4, 6, 6, 11, 20, 40, 45 is a 1, 5 underlying partition for 237 and 7.)

LEMMA 1. *There is a one-to-one correspondence between r, m underlying partitions for n and k and the partitions counted by $D_{r,m}(n)$ which have exactly k parts.*

PROOF. Let $a_1 \le a_2 \le \cdots \le a_k$ be an r, m underlying partition for n and k and, for $1 \le i \le k$, define $b_i = a_i + m(i - 1)$. Then $b_1 < b_2 < \cdots < b_k$ is a partition counted by $D_{r,m}(n)$, and this correspondence is uniquely reversible. (Example: 4, 6, 6, 11, 20, 40, 45 corresponds to 4, 11, 16, 26, 40, 65, 75.)

DEFINITION. An ordering of the parts in an r, m underlying partition for n and k, say $(a_1, a_2, \ldots, a_k)$, is called an Ω-*ordering* if the following inequalities are satisfied for any i and j, $1 \le i < j \le k$.

(1) $a_i \equiv a_j \equiv 0 \pmod m \Rightarrow a_i < a_j$;

(2) $a_i \equiv \pm a_j \equiv \pm r \pmod m \Rightarrow a_i \le a_j$;

(3)
$$\left.\begin{array}{l} a_i \equiv \pm r \pmod m \\ a_j \equiv 0 \pmod m \end{array}\right\} \Rightarrow a_i + mi \le \left\lfloor \frac{a_j + mi}{2} \right\rfloor_r ;$$

(4)
$$\left.\begin{array}{l} a_i \equiv 0 \pmod m \\ a_j \equiv \pm r \pmod m \end{array}\right\} \Rightarrow \left\lfloor \frac{a_i + mi}{2} \right\rfloor_r < a_j + mi,$$

or equivalently,

$$\left\lceil \frac{a_i + m(i - 1)}{2} \right\rceil_r < a_j + mi.$$

In this definition, $\lfloor A \rfloor_r$ (resp. $\lceil A \rceil_r$) is the greatest integer less than or equal to A (resp. least integer greater than or equal to A) and congruent to $\pm r \pmod m$.

LEMMA 2. *Every underlying partition has a unique Ω-ordering.*

PROOF. Given an r, m underlying partition for n and k, let $b_1 \le b_2 \le \cdots b_{k'}$ be the parts congruent to $\pm r \pmod m$ and let $c_1 < c_2 < \cdots < c_{k''}$ be the parts divisible by m. If $(a_1, a_2, \ldots, a_k)$ is an Ω-ordering of this partition, then a_1 equals

either b_1 or c_1. By properties (3) and (4) of an Ω-ordering, if $b_1 + m \leqslant \lfloor (c_1 + m)/2 \rfloor_r$, then $a_1 = b_1$. If $b_1 + m > \lfloor (c_1 + m)/2 \rfloor_r$, then $a_1 = c_1$. Thus a_1 is uniquely determined.

We proceed inductively, and assume that $a_1, a_2, \ldots, a_{j-1}$ have been uniquely determined, and that we have used the parts $b_1, \ldots, b_{s-1}, c_1, \ldots, c_{t-1}$. Thus a_j is either b_s or c_t. Again by properties (3) and (4), if $b_s + mj \leqslant \lfloor (c_t + mj)/2 \rfloor_r$, then $a_j = b_s$. If $b_s + mj > \lfloor (c_t + mj)/2 \rfloor_r$, then $a_j = c_t$. Thus a_j is uniquely determined, and Lemma 2 is proved. (Example: The Ω-ordering of 4, 6, 6, 11, 20, 40, 45 is 4, 20, 6, 6, 40, 45, 11.)

DEFINITION. Given a partition, $a_1 < a_2 < \cdots < a_p$, counted by $C_{r,m}(n)$, we subdivide it, working from left to right, into blocks of at most two parts such that if $a_j + m > a_{j+1}$ and if (1) $j = 1$, or (2) $a_j > a_{j-1} + m$, or (3) a_{j-2} and a_{j-1} share a block, then a_j, a_{j+1} share a block. Otherwise, a_j inhabits a block by itself. The *order* of the partition is the number of such blocks. (Example: The partition 4, 11, 14, 16, 21, 29, 31, 34, 36, 41 counted by $C_{1,5}(237)$ is blocked as follows 4|11, 14|16|21|29, 31|34, 36|41. It has order 7.)

LEMMA 3. *There is a one-to-one correspondence between r, m underlying partitions for n and k and the partitions counted by $C_{r,m}(n)$ which have order k.*

PROOF. Let $(a_1, \ldots, a_k)$ be an r, m underlying partition for n and k with the Ω-ordering. Define a partition $b_1, \ldots, b_k$ of n by setting $b_i = a_i + m(i - 1)$ for $1 \leqslant i \leqslant k$. Each b_i which is divisible by m is replaced by the pair $\lfloor b_i/2 \rfloor_r, \lceil b_i/2 \rceil_r$. From property (3) of the Ω-ordering, we have that if $i < j$, $b_i \equiv \pm r \pmod{m}$ and $b_j \equiv 0 \pmod{m}$, then $b_i + m \leqslant \lfloor b_j/2 \rfloor_r$. From property (4), we have that if $i < j$, $b_i \equiv 0 \pmod{m}$ and $b_j \equiv \pm r \pmod{m}$, then $\lceil b_i/2 \rceil_r < b_j$. Thus the resulting partition is one counted by $C_{r,m}(n)$ with order k. This procedure is uniquely reversed if parts sharing the same block are first added together, and then $m(i - 1)$ is subtracted from the ith part as read from the left. (Example: 4, 20, 6, 6, 40, 45, 11 becomes 4, 25, 16, 21, 60, 70, 41 which becomes 4|11, 14|16|21|29, 31|34, 36|41.)

Lemmas 1 and 3, taken together, give the desired correspondence which establishes Schur's Theorem.

REFERENCES

1. G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers*, 4th ed., Clarendon Press, Oxford, 1960.
2. I. J. Schur, *Zur additiven Zahlentheorie*, S.-B. Preuss. Akad. Wiss. Phys.-Math. Kl., 1926, pp. 488–495. (Reprinted in I. Schur, *Gesammelte Abhandlungen*, vol. 3, Springer, Berlin, 1973, pp. 43–50.)

DEPARTMENT OF MATHEMATICS, PENNSYLVANIA STATE UNIVERSITY, UNIVERSITY PARK, PENNSYLVANIA 16802

Current address: School of Mathematics, Institute for Advanced Study, Princeton, New Jersey 08540