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A REMARK ON COSINE FAMILIES

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ABSTRACT. Let $C(t)$, $t \in R$, be a strongly continuous cosine family and A its infinitesimal generator. Then the set $E \overset{\text{def}}{=} \{x \in X\colon C(t)x$ is once continuously differentiable in t on $R\}$ of the Banach space X is contained in the domain of $(-A)^\alpha$ for $0 \leqslant \alpha < 1/2$.

The purpose of this note is to prove for a strongly continuous cosine family $C(t)$, $t \in R$, defined on a Banach space X and with infinitesimal generator A, that the set $E \overset{\text{def}}{=} \{x \in X\colon C(t)x$ is once continuously differentiable in t on $R\}$ is contained in the set $D[(-A)^\alpha]$, $0 \leqslant \alpha < 1/2$. The set $D[(-A)^\alpha]$ is the domain of the α power of the operator $-A$.

A one parameter family $C(t)$, $t \in R$, of bounded linear operators mapping the Banach space X into itself is called a strongly continuous cosine family if and only if

$C(t+s) + C(t-s) = 2C(t)C(s)$ for all $s, t \in R$,

$C(0) = I$,

$C(t)x$ is continuous in $t \in R$ for each fixed $x \in X$.

The associated sine family is given by $S(t)x = \int_0^t C(s)x\, ds$ for $x \in X$ and $t \in R$. The linear operator $A\colon X \to X$ defined by $Ax = C''(0)x$ and with dense domain $D(A) = \{x \in X\colon C(t)x$ is twice continuously differentiable in t on $R\}$ is called the infinitesimal generator of $C(t)$, $t \in R$. For other properties of cosine families used in this paper see **[2]** or **[9]**.

The following theorem appears in C. Travis and G. Webb **[8]** and **[9]**:

THEOREM 1. *Let $C(t)$, $t \in R$, be a strongly continuous cosine family with associated sine family $S(t)$, $t \in R$, and infinitesimal generator A. The following statements are equivalent*:

(i) *there exists a closed linear operator B on X such that $B^2 = A$ and B commutes with every operator in $B(X, X)$ which commutes with A; $S(t)$ maps X into $D(B)$ for each $t \in R$; $BS(t)x$ is continuous in $t \in R$ for each fixed $x \in X$;*

(ii) *$E = D(B)$, the domain of B.*

The conditions stated in part (i) of Theorem 1 have also been considered by H. Fattorini in **[2]** and **[3]**. Fattorini has shown in **[3]** that every strongly continuous cosine family defined on the Banach space L^p, $1 < p < \infty$, satisfies condition (i).

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However, there are cosine families which do not satisfy condition (i) as shown by J. Kisynski [4] and B. Nagy [6]. It remains open whether or not $E \subset D[(-A)^{1/2}]$ in general.

Proposition. *Let A be the infinitesimal generator of the strongly continuous cosine family $C(t)$, $t \in R$, such that there exists an $M > 0$ with $\|(\lambda - A)^{-1}\| \leqslant M/\lambda$ for all $\lambda > 0$. Then the set E is contained in $D[(-A)^\alpha]$ for $0 \leqslant \alpha < 1/2$.*

Proof. It is known, V. Balakrishman [1], that $-(-A)^{1/2}$ generates an analytic semigroup $T(t)$ with $T(t)X \subset D(A)$. Now if $x \in E$ then $S(t)x$ is twice continuously differentiable and the function

$$u(t) = \int_0^t T(t - v)S(v)x\,dv$$

solves the differential equation

$$u'(t) = -(-A)^{1/2}u(t) + S(t)x, \quad u(0) = 0.$$

Following a change of variable in the integral, we have

$$u'(t) = \int_0^t T(v)C(t - v)x\,dv$$

and

$$\begin{aligned} u''(t) &= \int_0^t T(v)AS(t - v)x\,dv + T(t)x = \int_0^t T(t - v)AS(v)x\,dv + T(t)x \\ &= Au(t) + T(t)x = -(-A)^{1/2}\int_0^t T(t - v)C(v)x\,dv + C(t)x \\ &= -(-A)^{1/2}u'(t) + C(t)x, \qquad t > 0. \end{aligned}$$

We can now write for $t > 0$

$$\begin{aligned} Au(t) + T(t)x &= -(-A)^{1/2}\left[-(-A)^{1/2}u(t) + S(t)x\right] + C(t)x \\ &= -Au(t) - (-A)^{1/2}S(t)x + C(t)x, \end{aligned}$$

which implies that

$$C(t)x = 2Au(t) + T(t)x + (-A)^{1/2}S(t)x \tag{$*$}$$

for all $x \in E$ and $t > 0$.

Under the stated conditions the fractional power of the operator $(-A)^{1/2}$ exist and $\|[(-A)^{1/2}]^\beta T(t)\| \leqslant C/t^\beta$ for all $t > 0$, for some $C > 0$, and for $0 < \beta < 1$. (See the book by S. Krein [5] or the lecture notes by A. Pazy [7].)

Therefore since $\int_0^t [(-A)^{1/2}]^\beta T(t - v)AS(v)x\,dv$ $(t > 0)$ exists for any $0 \leqslant \beta < 1$, we have that $Au(t) = \int_0^t T(t - v)AS(v)x\,dv$ is in $D[(-A)^{1/2}]^\beta$, $t > 0$, for all $0 \leqslant \beta < 1$. This is equivalent to $Au(t) \in D[(-A)^\alpha]$, $t < 0$, for all $0 \leqslant \alpha < 1/2$.

Appealing now to equation $(*)$, the fact that $(-A)^{1/2}S(t)x \in D[(-A)^{1/2}]$, $T(t)x \in D(A)$, and $Au(t) \in D[(-A)^\alpha]$ $(0 \leqslant \alpha < 1/2)$, for all $t > 0$, we have that $C(t)x \in D[(-A)^\alpha]$ for $t > 0$. The identity $x = 2C(t)C(t)x - C(2t)x$ gives the result, since $C(t)$, $t \in R$, leaves $D[(-A)^\alpha]$ invariant.

REMARK. The condition $\|(\lambda - A)^{-1}\| \leq M/\lambda$ for $\lambda > 0$ is not as restrictive as it may seem since by Theorem 2.7 of [9], $\|(\lambda^2 - A)^{-1}\| \leq M/\lambda(\lambda - w)$ $(\lambda > w)$. If A generates a strongly continuous cosine family then so does $A_b = A - b^2 I$ [2]. Thus if one chooses $b \geq w$, we have for all $\lambda > 0$

$$\|(\lambda^2 - A_b)^{-1}\| = \|(\lambda^2 + b^2 - A)^{-1}\| \leq \frac{M}{\lambda(\lambda^2 + b^2 - w)} \leq \frac{M'}{\lambda^2}.$$

Substituting $\sqrt{\lambda}$ for λ the condition of the proposition is satisfied by A_b. In this sense (after a suitable translation) every strongly continuous cosine family satisfies the above proposition.

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