PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 79, Number 3, July 1980

IDEMPOTENT MULTIPLIERS ON SPACES OF CONTINUOUS FUNCTIONS WITH p-SUMMABLE FOURIER TRANSFORMS

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ABSTRACT. Let G denote a compact abelian group, and A^p the space of functions continuous on G and having p-summable Fourier transforms. The idempotent multipliers from A^p to A^q are characterised for $p, q \in [1, 2]$.

Throughout G will denote a compact Hausdorff abelian group with character group Γ. Given $p \in [1, 2]$ we shall write A^p for the Banach space of functions continuous on G with p-summable Fourier transforms, and normed by $\|f\| = \|f\|_\infty + \|\hat{f}\|_p$. Note that we can identify A^2 with C and A^1 with A, the spaces of functions on G that are continuous and have absolutely convergent Fourier series respectively. An application of the Hahn-Banach theorem shows that the dual $(A^p)'$ of A^p can be identified with $M + Fl^{p'}$, where M is the space of Radon measures on G and $Fl^{p'}$ is the space of pseudomeasures on G with Fourier transforms in $l^{p'}$. Here p' is the exponent conjugate to p, that is, $p' = p/(p-1)$ with the usual convention if $p = 1$. The duality is expressed by

$$h(f) = \mu * f(0) + \sigma * f(0), \qquad f \in A^p,$$

where $h \in (A^p)'$, $\mu \in M$ and $\sigma \in Fl^{p'}$.

We shall write (A^p, A^q) for the set of pseudomeasures that are multipliers from A^p to A^q; that is, $\sigma \in (A^p, A^q)$ if and only if for every $f \in A^p$ there exists $g \in A^q$ with $\sigma * f = g$. Using the above characterisation of the dual of A^p the following result can be proved.

LEMMA 1. *Given* $p \in [1, 2]$ *and* $q \in [p, 2]$,

$$(A^p, A^q) = (A^p, A^p) = (A^p, C) = M + Fl^{p'}.$$

The A^p spaces and their multiplier spaces have been considered previously; see [1], [2], [5] and [6]. Here we are interested in characterising the set of multipliers $\phi \in (A^p, A^q)$ that are idempotent; that is, satisfy $\phi * \phi = \phi$. We shall write $(A^p, A^q)_I$ for the set of all idempotent elements of (A^p, A^q). In addition, we shall denote by J the set of all idempotent measures on G, by ξ_Ξ the characteristic function of the set Ξ, and by $|\Xi|$ the cardinality of Ξ if Ξ is finite.

By considering Fourier transforms we see that to each $\phi \in (A^p, A^q)_I$ there corresponds unique $\Xi \subseteq \Gamma$ for which $\hat{\phi} = \xi_\Xi$. In the case $p = 1$, $q \in [1, 2]$, it is easy to see (using Lemma 1) that

Received by the editors October 20, 1978.
AMS (MOS) subject classifications (1970). Primary 43A22, 43A15.

$$(A, A^q)_I = (A, A)_I = \{\phi: \hat{\phi} = \xi_\Xi \text{ for some } \Xi \subseteq \Gamma\}.$$

For other values of p, q, $p < q$, the idempotent multipliers are precisely the idempotent measures.

THEOREM 1. *For $p \in (1, 2]$, $q \in [p, 2]$, $(A^p, A^q)_I = (A^p, A^p)_I = J$.*

PROOF. The first equality follows directly from Lemma 1. As for the second, consider $\phi \in (A^p, A^p)_I$. By Lemma 1 we can write $\phi = \mu - \sigma$, where $\mu \in M$ and $\hat{\sigma} \in l^{p'}$. Choose a positive integer n satisfying $2n < p' < 2(n+1)$. Then $\hat{\sigma}^{n+1} \in l^2$ and hence $\sigma^{*r} \in L^2$ for all $r \geq n+1$, where σ^{*r} denotes the r-fold convolution of σ with itself. Now

$$\mu^{*s} = (\phi + \sigma)^{*s} = \sum_{r=0}^{s} \binom{s}{r} \phi^{*s-r} * \sigma^{*r} = \sum_{r=0}^{s-1} \binom{s}{r} \phi * \sigma^{*r} + \sigma^{*s}$$

(recall that $\phi^{*k} = \phi$ for all $k \geq 1$) and consideration of this expression for $s = n+1, n+2, \ldots, 2n+1$ gives

$$\begin{bmatrix} 1 & \binom{n+1}{1} & \cdots & \binom{n+1}{n} \\ 1 & \binom{n+2}{1} & \cdots & \binom{n+2}{n} \\ \vdots & \vdots & & \vdots \\ 1 & \binom{2n+1}{1} & \cdots & \binom{2n+1}{n} \end{bmatrix} \begin{bmatrix} \phi \\ \phi * \sigma \\ \vdots \\ \phi * \sigma^{*n} \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_{n+1} \end{bmatrix}$$

where $\mu_i \in M$, $i = 1, 2, \ldots, n+1$. The coefficient matrix is nonsingular; indeed using the identity $\binom{t+1}{s} - \binom{t}{s} = \binom{t}{s-1}$, $t \geq s$, it can be reduced to the upper triangular matrix (α_{ij}), where $\alpha_{ij} = \binom{n+1}{j-i}$ for $i \leq j$. It follows that $\phi \in M$ and hence $\phi \in J$, thus showing that $(A^p, A^p)_I \subseteq J$. The reverse inclusion is clear, completing the proof of the theorem. □

For $p \in (1, 2]$, $q \in [1, p)$ the only idempotent multipliers ϕ from A^p to A^q are those given by $\hat{\phi} = \xi_\Xi$, where Ξ is finite. To prove this we require some preliminary results.

Let $\mathcal{K}$ denote the family of all cosets in Γ and $\mathcal{R}$ the Boolean ring generated by $\mathcal{K}$. Every $\Xi \in \mathcal{R}$ can be written as a finite union of sets of the form $\Pi = \bigcap_{i=1}^{n} \Lambda_i^c \cap \bigcap_{i=n+1}^{r} \Lambda_i$, where $\Lambda_i \in \mathcal{K}$ and Λ_i^c denotes the complement of Λ_i. If Ξ is infinite then so must be at least one such set Π.

LEMMA 2. *Given an infinite set Π of the above type there exists a finite set Υ with the property that $\Pi \cup \Upsilon$ contains an infinite coset.*

PROOF. First we show that a finite intersection of cosets is either empty or itself a coset. Indeed consider $\Lambda_1, \Lambda_2 \in \mathcal{K}$ such that $\Lambda_1 \cap \Lambda_2$ is nonempty. Then $\Lambda_1 = \eta_1 + \Omega_1$, $\Lambda_2 = \eta_2 + \Omega_2$, where Ω_1, Ω_2 are subgroups of Γ, and there exist $\chi_1 \in \Omega_1$, $\chi_2 \in \Omega_2$ such that $\eta_1 + \chi_1 = \eta_2 + \chi_2$. Hence

$$\Lambda_1 \cap \Lambda_2 = (\eta_2 + \chi_2 + \Omega_1) \cap (\eta_2 + \chi_2 + \Omega_2) = \eta_2 + \chi_2 + \Omega_1 \cap \Omega_2 \in \mathcal{K}.$$

Thus a nonempty intersection of two cosets is itself a coset and the result extends to arbitrary finite intersections by induction.

Applying this to (the nonempty set) Π we see that

$$\Pi = \bigcap_{i=1}^{n} \Lambda_i^c \cap \Lambda, \qquad \Lambda_i, \Lambda \in \mathcal{K}.$$

By relabelling if necessary we can assume that $\Pi \cup \Upsilon$ contains the infinite set $\bigcap_{i=1}^{n} \Lambda_i^c \cap \Lambda$, where Υ is finite and each Λ_i is infinite. Now $\Lambda_i = \eta_i + \Omega_i$, $\Lambda = \eta + \Omega$ for some subgroups Ω_i, Ω of Γ. We consider the following cases:

(1) $\bigcap_{i=1}^{n} \Omega_i \cap \Omega$ is infinite. Then $\Pi \cup \Upsilon$ contains $\bigcap_{i=1}^{n} \Lambda_i' \cap \Lambda$ which, for suitable cosets Λ_i' of Ω_i, is nonempty; just use the fact that each Λ_i^c is a union of cosets of Ω_i. By the argument used in the first part of the proof of the lemma, $\bigcap_{i=1}^{n} \Lambda_i' \cap \Lambda$ is an infinite coset in Γ.

(2) $\bigcap_{i=1}^{n} \Omega_i \cap \Omega$ is finite, $\bigcap_{i \in I} \Omega_i \cap \Omega$ is infinite for some $I \subseteq \{1, 2, \ldots, n\}$ with $|I| = n - 1$. Then there are cosets Λ_i'' of Ω_i such that $\Pi \cup \Upsilon$ contains the nonempty set $\bigcap_{i \in I} \Lambda_i'' \cap \Lambda \cap \Lambda_j^c$, where $\{j\} = \{1, 2, \ldots, n\} \setminus I$, and $\Upsilon' = \bigcap_{i \in I} \Lambda_i'' \cap \Lambda \cap \Lambda_j$ is (empty or) finite. Thus $\Pi \cup \Upsilon \cup \Upsilon' \supseteq \bigcap_{i \in I} \Lambda_i'' \cap \Lambda$, an infinite coset.

(3) $\bigcap_{i \in I} \Omega_i \cap \Omega$ is finite for all $I \subseteq \{1, 2, \ldots, n\}$ with $|I| = n - 1$. In this case we return to (2) above and repeat the process.

After $n - 1$ steps we have that $\Omega_i \cap \Omega$ is finite for each $i \in \{1, 2, \ldots, n\}$ and

$$\Pi \cup \Upsilon \cup \bigcup_{i=1}^{n} (\Lambda_i \cap \Lambda) \supseteq \left(\bigcap_{i=1}^{n} \Lambda_i^c \cap \Lambda \right) \cup \left(\bigcup_{i=1}^{n} \Lambda_i \cap \Lambda \right) = \Lambda$$

is the required infinite coset. $\square$

THEOREM 2. *Let $\Xi \in \mathcal{R}$ be infinite and $p \in (1, 2]$. Then there exists a continuous function f with* $\operatorname{supp} \hat{f} \subseteq \Xi$ *and $\hat{f} \in l^p \setminus l^q$ for all $q \in [1, p)$.*

PROOF. By Lemma 2 we have the existence of a finite set Υ such that $\Xi \cup \Upsilon$ contains an infinite coset Λ. We shall construct f as in the statement of the theorem with $\operatorname{supp} \hat{f} \subseteq \Lambda$, and then the result will follow. There are two cases:

(a) The subgroup $\Omega = -\eta + \Lambda$ has an element χ of infinite order. Now construct Rudin-Shapiro polynomials f_n in the usual way ([7, (37.19)]) so that $\hat{f}_n$ takes values in $\{-1, 0, 1\}$,

$$\operatorname{supp} \hat{f}_n = \{0, \chi, \ldots, (2^n - 1)\chi\} \quad \text{and} \quad \|f_n\|_\infty < 2^{(n+1)/2}.$$

Choose $\chi_n \in \Omega$ so that the spectra of the functions $\chi_n f_n$ are pairwise disjoint and put $f = \sum_{n=0}^{\infty} (n+1)^{-2} 2^{-n/p} \eta \chi_n f_n$. Then

$$\sum_{n=0}^{\infty} (n+1)^{-2} 2^{-n/p} \|\eta \chi_n f_n\|_\infty < \infty,$$

so that $f \in C$. Clearly $\operatorname{supp} \hat{f} \subseteq \Lambda$. That $f \in A^p$ follows from the estimate

$$\|\hat{f}\|_p < \sum_{n=0}^{\infty} (n+1)^{-2} 2^{-n/p} 2^{n/p} < \infty,$$

whereas $\|\hat{f}\|_q^q = \sum_{n=0}^{\infty} (n+1)^{-2q} 2^{-nq/p} 2^n$ shows that $f \notin A^q$ for any $q < p$.

(b) $-\eta + \Lambda = \Omega$ is a torsion subgroup of Γ, in which case there is a strictly increasing sequence (Ω_n) of finite subgroups of Ω. For each n construct f_n (as in [4]) with the properties $|\hat{f}_n| = 1$ on Ω_n, $\operatorname{supp} \hat{f}_n = \Omega_n$ and $\|f_n\|_\infty = |\Omega_n|^{1/2}$. As in (a) choose $\chi_n \in \Omega$ so that the spectra of the functions $\chi_n f_n$ are pairwise disjoint and put $f = \sum_{n=0}^{\infty}(n+1)^{-2}2^{-n/p}\eta\chi_n f_n$. $\square$

THEOREM 3. *Let $p \in (1, 2]$, $q \in [1, p)$ be given. Then $\phi \in (A^p, A^q)_I$ if and only if $\hat{\phi} = \xi_\Xi$ for some finite set $\Xi \subseteq \Gamma$.*

PROOF. Sufficiency is obvious. As for the other direction, consider $\phi \in (A^p, A^q)_I$. Using $(A^p, A^q)_I \subseteq (A^p, A^p)_I$ we see from Theorem 1 that $\phi \in J$ and hence, by [3, Theorem 3], $\hat{\phi} = \xi_\Xi$ for some set Ξ belonging to the coset ring $\mathfrak{R}$ of Γ. Suppose now that Ξ is infinite. By Theorem 2 there exists $f \in C$ with $\operatorname{supp} \hat{f} \subseteq \Xi$ and $\hat{f} \in l^p \setminus l^q$. This contradicts the assumption that $\phi \in (A^p, A^q)_I$. $\square$

COROLLARY. *Suppose G is infinite and let $p \in (1, 2]$, $q \in [1, p)$ be given. Then $(A^p, A^q)_I \subsetneq (A^p, A^p)_I$.*

PROOF. That this inclusion is strict follows from Theorem 3 together with the fact that δ_0 (the Dirac measure at 0) belongs to $(A^p, A^p)_I$. $\square$

ACKNOWLEDGEMENT. We should like to thank Dr. John F. Price for suggesting some of the problems considered in this paper.

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