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A SHORT PROOF FOR A.E. CONVERGENCE OF GENERALIZED CONDITIONAL EXPECTATIONS

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ABSTRACT. Let $L_s(\mu)$ be the space of real valued random variables with $\mu(|f|^s) < \infty$, $1 < s < \infty$. Let $C \subset L_s(\mu)$ be a closed convex set. For each $f \in L_s(\mu)$ there exists a unique element $\mu_s(f|C)$ with $\|f - \mu_s(f|C)\|_s < \|f - c\|_s$ for every $c \in C$. Let C_n be a decreasing or increasing sequence of closed convex lattices converging to the closed convex lattice C_∞. We show that $\mu_s(f|C_n) \to \mu_s(f|C_\infty)$ μ-a.e. for every $f \in L_s(\mu)$.

This result contains the results of a.e. convergence of prediction sequences of Ando-Amemiya and the result of Brunk and Johansen of a.e. convergence of conditional expectations given σ-lattices.

Let μ be a measure defined on a σ-algebra $\mathfrak{A}$ over Ω. For each s with $1 < s < \infty$ denote by $L_s(\mu)$ the space of equivalence classes of real valued random variables f with $\mu(|f|^s) < \infty$. Then $L_s(\mu)$ is a uniformly convex Banach space. Hence for each closed convex set C and each $f \in L_s(\mu)$ there exists a unique element $\mu_s(f|C)$ fulfilling

(i) $\mu_s(f|C) \in C$,

(ii) $\|f - \mu_s(f|C)\|_s < \|f - c\|_s$ for all $c \in C$,

where $\|f\|_s = [\mu(|f|^s)]^{1/s}$.

If μ is a probability measure P, $s = 2$, $\mathfrak{B} \subset \mathfrak{A}$ is a sub-σ-field and C is the system of all square integrable equivalence classes of functions which contain a $\mathfrak{B}$-measurable function, then $P_2(f|C)$ is the usual conditional expectation of f given $\mathfrak{B}$, i.e. $P_2(f|C) = P^{\mathfrak{B}}f$.

If μ is a probability measure, $s \neq 2$, $\mathfrak{B} \subset \mathfrak{A}$ is a sub-σ-field and C is the system of all equivalence classes of functions of $L_s(P)$ which contain a $\mathfrak{B}$-measurable function, then $P_s(f|C)$ is the s-prediction $P_s^{\mathfrak{B}}f$ of Ando-Amemiya [1].

If $s = 2$, $\mathfrak{C} \subset \mathfrak{A}$ is a sub-σ-lattice and C is the system of all square integrable equivalence classes of functions which contain a $\mathfrak{C}$-measurable function, then $\mu_2(f|C)$ is the conditional expectation $\mu(f|\mathfrak{C})$ of f given $\mathfrak{C}$ in the sense of [2].

Let C_n be a decreasing or increasing sequence of closed convex subsets of $L_s(\mu)$. Put $C_\infty = \bigcap_{n \in \mathbb{N}} C_n$ for the decreasing case and let C_∞ be the closure of $\bigcup_{n \in \mathbb{N}} C_n$ with respect to $\| \ \|_s$ for the increasing case.

Using the definition (i), (ii) of $\mu_s(f|C)$ one directly obtains

$$\|f - \mu_s(f|C_n)\|_s \to \|f - \mu_s(f|C_\infty)\|_s. \tag{$*$}$$

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The following theorem gives a very short proof for the μ-a.e. convergence of $\mu_s(f|C_n)$ to $\mu_s(f|C_\infty)$. The proof is even for the classical conditional expectation $P^{\mathfrak{B}}f$ shorter and more transparent than the known ones. It contains the a.e. convergence result given in [1] and [2].

C is a lattice if $f, g \in C$ imply $f \vee g$, $f \wedge g \in C$ where $f \vee g$ $[f \wedge g]$ is the pointwise maximum [minimum] of f and g.

If C_n is a decreasing or increasing sequence of closed convex lattices then C_∞ is a closed convex lattice too.

THEOREM. *Let $1 < s < \infty$ and $C_n \subset L_s(\mu)$, $n \in \mathbf{N}$, be an increasing or decreasing sequence of closed convex lattices converging to the closed convex lattice C_∞. Then*

$$\mu_s(f|C_n) \to \mu_s(f|C_\infty) \quad \mu\text{-a.e.} \quad \text{for every } f \in L_s(\mu).$$

PROOF. Let $f, g, h \in L_s(\mu)$. Using the trivial identity

$$|f(\omega) - g(\omega) \wedge h(\omega)|^s + |f(\omega) - g(\omega) \vee h(\omega)|^s = |f(\omega) - g(\omega)|^s + |f(\omega) - h(\omega)|^s$$

we obtain

$$\int |f - g \wedge h|^s \, d\mu + \int |f - g \vee h|^s \, d\mu = \int |f - g|^s \, d\mu + \int |f - h|^s \, d\mu. \tag{1}$$

Let C_n be increasing. We shall show that for $n < m$

$$\|f - \mu_s(f|C_n) \wedge \cdots \wedge \mu_s(f|C_m)\|_s \leqslant \|f - \mu_s(f|C_n) \wedge \cdots \wedge \mu_s(f|C_{m-1})\|_s. \tag{2}$$

We apply (1) to $g = \mu_s(f|C_n) \wedge \cdots \wedge \mu_s(f|C_{m-1})$, $h = \mu_s(f|C_m)$. If (2) would be false, (1) implies

$$\|f - g \vee h\|_s = \|f - [\mu_s(f|C_n) \wedge \cdots \wedge \mu_s(f|C_{m-1})] \vee \mu_s(f|C_m)\|_s < \|f - \mu_s(f|C_m)\|_s$$

which contradicts

$$[\mu_s(f|C_n) \wedge \cdots \wedge \mu_s(f|C_{m-1})] \vee \mu_s(f|C_m) \in C_m.$$

From (2) we obtain for all $n < m$

$$\|f - \mu_s(f|C_n) \wedge \cdots \wedge \mu_s(f|C_m)\|_s \leqslant \|f - \mu_s(f|C_n)\|_s. \tag{3}$$

Using the lemma of Fatou ($m \to \infty$), (3) implies that for all n

$$\Big\|f - \bigwedge_{m \geqslant n} \mu_s(f|C_m)\Big\|_s \leqslant \|f - \mu_s(f|C_n)\|_s. \tag{4}$$

Applying once more the lemma of Fatou ($n \to \infty$), (4) and (*) imply

$$\Big\|f - \underline{\lim}_{n \in \mathbf{N}} \mu_s(f|C_n)\Big\|_s \leqslant \underline{\lim}_{n \in \mathbf{N}} \|f - \mu_s(f|C_n)\|_s = \|f - \mu_s(f|C_\infty)\|_s. \tag{5}$$

Since $\mu_s(f|C_n) \wedge \cdots \wedge \mu_s(f|C_m) \in C_m \subset C_\infty$ and C_∞ is $\|\ \|_s$-closed, (4) and (5) imply that

$$\underline{\lim}_{n \in \mathbf{N}} \mu_s(f|C_n) \in C_\infty.$$

Hence

$$\underline{\lim}_{n\in\mathbf{N}} \mu_s(f|C_n) = \mu_s(f|C_\infty).$$

Using

$$\|f - \mu_s(f|C_n) \vee \cdots \vee \mu_s(f|C_m)\|_s < \|f - \mu_s(f|C_n) \vee \cdots \vee \mu_s(f|C_{m-1})\|_s, \qquad (2)^*$$

which follows again from (1), one obtains completely analogously $\overline{\lim}_{n\in\mathbf{N}} \mu_s(f|C_n) = \mu_s(f|C_\infty)$, which yields the assertion for the increasing case. If C_n is decreasing one obtains from (1) [instead of (2) and (2)*] for $n < m$:

$$\|f - \mu_s(f|C_n) \wedge \cdots \wedge \mu_s(f|C_m)\|_s < \|f - \mu_s(f|C_{n+1}) \wedge \cdots \wedge \mu_s(f|C_m)\|_s,$$
$$\|f - \mu_s(f|C_n) \vee \cdots \vee \mu_s(f|C_m)\|_s < \|f - \mu_s(f|C_{n+1}) \vee \cdots \vee \mu_s(f|C_m)\|_s. \qquad (6)$$

From (6) we obtain the assertion completely analogously as in the increasing case.

References

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