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THE COBORDISM CLASSIFICATION OF HYPERSURFACES IN LENS SPACES

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ABSTRACT. In Theorem A we classify by cobordism type the codimension one submanifolds of lens spaces L_d^{2n+1} (quotient of S^{2n+1} by the action of the dth roots of unity). A related immersion result is also obtained.

Introduction. A classical generalization of the Jordon Curve Theorem asserts that every compact codimension one submanifold M of the n-sphere S^n separates. Further, since S^n is orientable, M is orientable. However, the sphere is not unique with this property, for if N is an orientable manifold such that $H^1(N; Z_2) = 0$ then any codimenison one submanifold separates and is orientable.

We study the situation for more general ambient manifolds in this paper and obtain in Theorem A the cobordism classification of codimension one submanifolds in lens spaces, partially generalizing results of Bredon and Wood for 3-dimensional lens spaces. The basic technique will apply to other classes of ambient manifolds as well. A partial extension to codimension one immersions is also given. In §1 we obtain some general results and state Theorem A, the proof of which is given in §2.

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1. Let N^n be a smooth connected closed n-dimensional manifold. It is known that every $x \in H_{n-1}(N; Z_2)$ can be realized as $x = i_*[M]$ where $[M]$ is the Z_2 orientation class of a closed, connected $(n-1)$-dimensional manifold M^{n-1}, and $M \xrightarrow{i} N$ is a smooth embedding. We call such an embedded manifold a *hypersurface in N* and say that it *carries* the element x; for convenience, a hypersurface is assumed connected. A well-known consequence of the duality theorems for manifolds asserts that $x = 0$ if and only if M separates N; that is, the complement of iM consists of two components. Thus if $x = 0$, M is nullbordant. In fact by [5], two hypersurfaces which carry a given $x \in H_{n-1}(N; Z_2)$ must be cobordant. This is clear if the two hypersurfaces can be made disjoint, however that is rarely possible. Note that the converse is not true: distinct elements of $H_{n-1}(N; Z_2)$ may be carried by cobordant manifolds (or even the same manifold). For instance,

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RP^{2n-1}, a nullbordant manifold, carries the nonzero element in $H_{2n-1}(RP^{2n}; Z_2)$, where RP^k is the k-dimensional real projective space. In any case, if one can identify the cobordism classes of hypersurfaces carrying elements of $H_{n-1}(N; Z_2)$, explicit necessary conditions are obtained for a given $(n-1)$-dimensional manifold to embed in N. For example, every hypersurface in RP^{2n} is nullbordant.

Further conditions can be obtained by considering the orientability of a hypersurface in N. Assume N orientable. Note that if M separates N then M is orientable, since M is the boundary of an orientable manifold. The converse of this is true if $H_{n-1}(N; Z_2) = 0$, by the universal coefficient theorem. In fact this may be seen as the exact meaning that $H_{n-1}(N; Z) = 0$, for any nonzero element is carried by the inverse image of a regular value of a smooth function from N to S^1, providing a nonseparating orientable hypersurface. We record these observations in the following proposition.

PROPOSITION 1.1. *Let N be an orientable smooth closed n-dimensional manifold. Then $H_{n-1}(N; Z) = 0$ if and only if for every hypersurface M in N the following are equivalent*:

(i) *M separates N,*

(ii) *M is orientable.*

Furthermore, $H_{n-1}(N; Z_2) = 0$ if and only if (i) *and* (ii) *are true for every hypersurface M in N.* □

Note that the proposition applies to any simply connected N, for then $H_{n-1}(N; Z) = 0$. As a corollary, if this condition holds any orientable hypersurface in N is nullbordant. In general of course, orientability and cobordism type are unrelated. We exploit this perhaps unexpected relationship to obtain in Theorem A below a classification by cobordism type of hypersurfaces in lens spaces.

The following proposition is a special case of Theorem A which is immediate from the considerations above without further analysis. As noted in the first paragraph, every hypersurface in RP^{2n} is nullbordant, hence this proposition completes the classification of hypersurfaces in all real projective spaces.

PROPOSITION 1.2. *Let M^{2n} be a compact connected submanifold of RP^{2n+1}. If M is orientable, M is nullbordant. Otherwise M is cobordant to RP^{2n}.*

PROOF. By Proposition 1.1, M nonorientable implies M does not separate RP^{2n+1}. Therefore M represents the nonzero class in $H_{2n}(RP^{2n+1}; Z_2)$ which is also carried by RP^{2n}. Thus by Thom [5], as noted in the introduction, M is cobordant to RP^{2n}. □

An immediate consequence is the following corollary due to Bredon and Wood [1]. An elementary geometric proof along with some historical information for this result is given in [3]. (We take this opportunity to correct our citation in [3] of the result of Bredon and Wood. The Klein bottle *does* embed in $N \times S^1$ if $N = S^2$: a great circle of $S^2 \times 0$, by "time" θ in $S^2 \times \theta$, is rotated through an angle θ (thanks to Daniel Asimov for providing this bottle).) Recall that the Klein bottle with handles is nullbordant.

COROLLARY. *The Klein bottle does not embed in RP^3. In fact, a surface embeds in RP^3 if and only if it is orientable or of odd Euler characteristic.*

Let L_d^{2n+1} be the quotient of S^{2n+1} defined by the coordinate-wise action in C^{n+1} of the dth roots of unity. Note that if d is odd, $H_{2n}(L_d^{2n+1}; Z_2) = 0$ and hence all hypersurfaces are both orientable and nullbordant. Of course, with $d = 2$ we just have Proposition 1.2.

THEOREM A. *Let $M^{2n} \xrightarrow{i} L_d^{2n+1}$ be a hypersurface and let $d = 2^r k$ with k odd. Then if M is orientable, M is nullbordant. If M is nonorientable, then d is even and,*

(i) *if $r = 1$, M is cobordant to RP^{2n},*

(ii) *if $r \geq 2$, and n is odd, M is nullbordant,*

(iii) *if $r \geq 2$, and n is even, M is cobordant to $RP^n \times RP^n$.* $\square$

The proof of Theorem A is given in §2. For 3-dimensional lens spaces Bredon and Wood in [1] prove a much deeper result: the nonorientable surface of genus s embeds in L_{2s}^3 and this is in fact the minimum possible genus. They obtain as well the minimum genus of a nonorientable surface in the more general lens spaces $L^3(p, q)$.

The following proposition shows how to obtain nonimmersion results from nonembedding results in the case of orientable manifolds and thus extends the applicability of the previous ideas. Here $\mathfrak{N}_*(N)$ is the unoriented cobordism group of Conner and Floyd [2, p. 18].

PROPOSITION 1.3. *If $M^{n-1} \xrightarrow{f} N^n$ is an immersion, M and N smooth closed orientable manifolds, then there is an orientable M_0^{n-1} and an embedding $M_0 \xrightarrow{f_0} N^n$ where f and f_0 represent the same element of $\mathfrak{N}_*(N)$.*

PROOF. Let $f_*[M] \in H_{n-1}(N; Z) \cong H^1(N; Z)$ be represented by a smooth map $N \to S^1$. Let M_0 be the inverse image of a regular value and f_0 the inclusion into N. Then f and f_0 represent the same element of $H_{n-1}(N; Z)$ hence of $H_{n-1}(N; Z)$. Then since the normal bundles are trivial, it is easy to see that they have equal Whitney numbers in the sense of [2, §17]. $\square$

COROLLARY. *If M^{2n} is orientable and immerses in L_d^{2n+1}, then M is nullbordant.* $\square$

In this manner one can see for example that there are no codimension one immersions of complex projective spaces CP^{2n}, $n > 1$, into lens spaces.

2. Corresponding to the various subgroups of the dth roots of unity we have the lattice of covering spaces

$$S^{2n+1} \to \cdots \to L_{d_i}^{2n+1} \to \cdots \to L_d^{2n+1}$$

where each d_i is a divisor of d. Thus if d is even, RP^{2n+1} is a covering space intermediate between L_d^{2n+1} and S^{2n+1}. The next proposition together with the results of §1 will then establish parts (i) and (ii) of Theorem A: lift M back to its preimage $\overline{M}$ in (i) S^{2n+1} or (ii) RP^{2n+1} respectively, and apply the proposition. Of course (i) also follows from Proposition 1.1.

PROPOSITION 2.1. *If $\overline{M} \xrightarrow{p} M$ is an odd degree covering projection, then $\overline{M}$ is cobordant to M.*

PROOF. Since $p_*[\overline{M}] = [M]$ in homology with Z_2 coefficients, it follows immediately from naturality that the Stiefel-Whitney numbers are equal. □

From here on we assume $d = 2^r$ with $r > 2$, for if $d = 2^r k$ and k is odd, we may lift M in L_d^{2n+1} back to $\overline{M}$ in $L_{2^r}^{2n+1}$ and apply Proposition 2.1 to the k-fold cover of M by $\overline{M}$. If $\overline{M}$ is not connected, choose some component of $\overline{M}$ which is an odd degree cover of M. Note that the case $r = 1$ is given in Proposition 1.2.

Proposition 2.2 below gives the Z_2 cohomology ring of $L_{2^r}^{2n+1}$. The additive structure of course is well known to be Z_2 in all degrees.

PROPOSITION 2.2. *$H^*(L_{2^r}^{2n+1}; Z_2)$, $r > 2$, is Z_2 in each degree generated by u, a of degrees one and two respectively, where $u^2 = 0$. Furthermore, the total Stiefel-Whitney class of $L_{2^r}^{2n+1}$ is $W(L) = (1 + a)^{n+1}$.*

PROOF. This result follows by induction on r from the following observations. Let $\overline{N} \xrightarrow{p} N$ be a connected 2-fold cover where the Z_2 cohomology of $\overline{N}$ and of N is (additively) Z_2 in every degree. Let $u \in H^1(N; Z_2)$ be the nonzero element. Then the Gysin sequence for p is

$$H^0(N) \xrightarrow{\cong} H^1(N) \xrightarrow{p^*} \cdots$$
$$1 \mapsto u$$
$$\cdots \to H^j(N) \xrightarrow{p^*} H^j(\overline{N}) \to H^j(N) \xrightarrow{\cup u} H^{j+1}(N) \to \cdots.$$

Thus p^* is zero in odd degrees and an isomorphism in even degrees. Further, if $q^*(x) = x \cup u$, q^* is also zero in odd degrees and an isomorphism in even degrees. In particular, $u^2 = 0$ and the odd degree Stiefel-Whitney classes of $\overline{N}$ are zero by naturality. Thus considering the composition of coverings

$$RP^{2n+1} \to L_4 \to \cdots \to L_{2^r} \to L_{2^{r+1}}$$

we deduce that $w_j(L_{2^r}) = w_j(RP^{2n+1})$ because p^* is an isomorphism for j even and because both classes are zero for j odd. □

Let $l = 2^j$ be the highest power of 2 which divides n. Then $W(L) = (1 + a)^n = (1 + a^l)^{\text{odd}} = 1 + a^l + \cdots$ (mod 2). Hence we have

LEMMA 2.1. *Let $\overline{w}_k$ denote the kth Stiefel-Whitney class of $L_{2^r}^{2n+1}$. If $\overline{w}_k \neq 0$, then $l = 2^j$ divides k and $\overline{w}_k = \overline{w}_l^m$ where $lm = k$.* □

LEMMA 2.2. *Let w_k be the kth Stiefel-Whitney class of a nonorientable manifold M^{2n} and let $u \in H^1(L_{2^r}^{2n+1}; Z_2)$ be the nonzero element. Let $M \xrightarrow{i} L_{2^r}^{2n+1}$ be an embedding. Then*

(i) $w_k = i^*(\overline{w}_{k-1}u)$ *if k is odd,*
(ii) $w_k = i^*(\overline{w}_k)$ *if k is even.*

PROOF. Since M is nonorientable, $i_*[M] \neq 0$ in Z_2 homology. The class $w_1(\nu(i)) = i^*u$ is Poincaré dual to $i_*[M]$. Thus $i^* W(L) = W(M) \cup (1 + i^*u)$. Since $u^2 = 0$, this gives $W(M) = i^*(W(L) \cup (1 + u))$. Hence, since $\bar{w}_k = 0$ for k odd,

$$w_k = i^*(\bar{w}_k + \bar{w}_{k-1}u) = \begin{cases} i^*\bar{w}_k, & k \text{ even}, \\ i^*(\bar{w}_{k-1}u), & k \text{ odd}. \end{cases} \quad \square$$

We now conclude the proof of Theorem A.

PROOF OF THEOREM A (iii). Assume n odd. Consider a nonzero Stiefel-Whitney number of M

$$0 \neq \langle w_1^{r_1} \cdots w_s^{r_s}, [M] \rangle = \langle i^*(\alpha u^{r_0}), [M] \rangle$$
$$= \langle \alpha u^{r_0}, u \cap [L] \rangle = \langle \alpha u^{r_0+1}, [L] \rangle.$$

Here α is a monomial in even degree $\bar{w}$'s given by Lemma 2.2(i). Since $u^2 = 0$, $r_0 = 0$, thus all w_k's here are of even degree. Thus this Stiefel-Whitney number is $\langle \bar{w}_{2^j}^m, u \cap [L] \rangle$ where $\bar{w}_{2^j}$ is the first nonzero Stiefel-Whitney class of L. Thus $m2^j = 2n$ so $m2^{j-1}$ is odd (since n is odd). But $j > 1$ since when n is odd $\bar{w}_2 = 0$ ($\bar{w}_2 = \binom{n+1}{1}$). This is a contradiction, hence all Stiefel-Whitney numbers of M are zero.

PROOF OF THEOREM A (iv). Assume n even. In this case as well, any monomial with an odd degree Stiefel-Whitney class will yield a zero Stiefel-Whitney number of M because of the introduction of $u^2 = 0$. Let us now establish this property for the Stiefel-Whitney numbers of $RP^n \times RP^n$.

Let $\bar{\omega}_k$, ω_k be the kth Stiefel-Whitney classes of $RP^n \times RP^n$ and RP^n respectively. Recall that $\bar{\omega}_k = \sum_{j=0}^{k} \omega_j \times \omega_{k-j}$ and $\omega_{k-j} \times \omega_j$ are either both zero or both nonzero. Thus if k is odd, a Stiefel-Whitney number having a monomial with $\bar{\omega}_k$ becomes twice the corresponding Stiefel-Whitney number with $\bar{\omega}_k$ replaced by $\sum_{j=0}^{(k-1)/2} \omega_j \times \omega_{k-j}$. Thus any Stiefel-Whitney number of $RP^n \times RP^n$ with an odd degree Stiefel-Whitney class will be zero. A similar, somewhat more involved argument will establish the equivalence of statements (1) and (2) below.

We complete the proof of Theorem A (iv) by noting successively that the statements (1)–(7) below are equivalent. For this we need to establish that i^* is one to one in even degrees. To see this note $(i^*a)^n \neq 0$ since $\langle i^*a^n, [M] \rangle = \langle a^n u, [L] \rangle \neq 0$ by Proposition 2.2. The following chain of equivalent statements completes the demonstration that M and $RP^n \times RP^n$ have the same Stiefel-Whitney numbers.

(1) $\langle \bar{\omega}_2^{r_1} \cdots \bar{\omega}_{2s}^{r_s}, [RP^n \times RP^n] \rangle \neq 0$,
(2) $\langle (\omega_1 \times \omega_1)^{r_1} \cdots (\omega_s \times \omega_s)^{r_s}, [RP^n \times RP^n] \rangle \neq 0$,
(3) $\omega_i \times \omega_i \neq 0$ for all i,
(4) $\omega_i \neq 0$ for all i,
(5) $\bar{w}_{2i}^{r_i} \neq 0$ for all i (a direct computation),
(6) $w_{r_1}^{r_i} \neq 0$ for all i (Lemma 2.3 since i^* is 1-1),
(7) $\langle w_2^{r_1} \cdots w_{2s}^{r_s}, [M] \rangle \neq 0$ (Lemma 2.2). $\square$

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