A CONSTRUCTION OF ERGODIC BMO FUNCTIONS

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ABSTRACT. One can explicitly construct unbounded functions having bounded mean oscillation (and in fact vanishing mean oscillation) with respect to any given ergodic measure-preserving transformation of a Lebesgue space.

1. BMO and VMO. A function which on the average does not differ too much from its average is said to have *bounded mean oscillation*, abbreviated BMO. When f is a real-valued function of a real variable, this condition is expressed by requiring that

$$\sup_I \frac{1}{m(I)} \int_I \left| f - \frac{1}{m(I)} \int_I f \, dm \right| dm < \infty, \tag{1.1}$$

where m is Lebesgue measure and the supremum is taken over all finite intervals $I \subset \mathbf{R}$ [9], [7] (see also [13] and the references given there). For the case of an ergodic measure-preserving transformation T of a probability space $(X, \mathcal{B}, \mu)$, an analogous definition has been given by Coifman and Weiss [3]: a measurable function f on X is in BMO if and only if

$$\|f\|_{\mathrm{BMO}} = \sup_m \operatorname{ess\,sup}_x \frac{1}{m} \sum_{k=0}^{m-1} \left| f(T^k x) - \frac{1}{m} \sum_{j=0}^{m-1} f(T^j x) \right| < \infty. \tag{1.2}$$

As stated in [3], these ergodic BMO functions have at least some of the interesting properties of the classical BMO functions: for example, they satisfy a certain distribution-function inequality, they can be represented as sums of bounded functions and their conjugates, and they form the dual of the space H^1 of all functions whose maximal functions are integrable.

Sarason [15] has defined the subclass of functions of *vanishing mean oscillation* (VMO); a real-valued function is in VMO if its oscillation over short intervals is small:

$$\lim_{\delta \to 0} \sup_{m(I) < \delta} \frac{1}{m(I)} \int_I \left| f - \frac{1}{m(I)} \int_I f \, dm \right| dm = 0.$$

For ergodic flows the analogous definition is formulated easily, but there is a problem for the case of a single transformation $T: X \to X$. However, Sarason proved that VMO coincides with the closure in BMO of the bounded uniformly continuous functions. Now because of the theorem of Jewett [8] and Krieger [11],

we may assume that (X, T) is a strictly ergodic subshift. That is, there is a closed shift-invariant subset Y of the compact metric space A^Z consisting of all bilateral sequences on a finite alphabet $A = \{a_1, a_2, \ldots, a_r\}$ such that (1) every orbit in Y is dense in Y, (2) there is a unique shift-invariant Borel probability measure ν on Y, and (3) $(X, \mathcal{B}, \mu, T)$ is measure-theoretically isomorphic to the system consisting of Y, the restriction to Y of the shift homeomorphism σ, and the measure ν. We assume then that $(X, \mathcal{B}, \mu, T) = (Y, \mathcal{B}_Y, \nu, \sigma)$, where $\mathcal{B}_Y$ is the Borel field of Y. Once a topology for X is produced in this way, we may form the BMO closure of $\mathcal{C}(X)$, thus producing an ergodic version of VMO.

2. Characterization and construction of ergodic VMO functions. We can characterize the ergodic VMO space $cl_{BMO}\, \mathcal{C}(X)$ by means of the Cesàro averages

$$M_n f(x) = \frac{1}{n} \sum_{k=0}^{n-1} f(T^k x).$$

Since (X, T) is strictly ergodic, $M_n f \to \int f\, d\mu$ uniformly on X for each $f \in \mathcal{C}(X)$. Write $f \overset{\cdot}{\in} \mathcal{C}(X)$ in case f coincides a.e. $d\mu$ with a continuous function on X, and define

$$\mathfrak{M}_0 = \left\{ f \in L^1(X, \mu): f - fT \overset{\cdot}{\in} \mathcal{C}(X) \text{ and } \left\| M_n f - \int f\, d\mu \right\|_{BMO} \to 0 \right\}$$

and

$$\mathfrak{M} = \left\{ f \in L^1(X, \mu): \left\| M_n f - \int f\, d\mu \right\|_{BMO} \to 0 \right\}.$$

Recall (see [5]) that the *Hilbert transform* $\tilde{f}(x)$ of a function $f \in L^1(X, \mu)$ is defined by

$$\tilde{f}(x) = \lim_{n \to \infty} \sum_{\substack{k=-n \\ k \neq 0}}^{n} \frac{f(T^k x)}{k},$$

the limit existing a.e. Let $\widetilde{\mathcal{C}(X)} = \{ \tilde{f}: f \in \mathcal{C}(X)\}$.

THEOREM 1. $\mathcal{C}(X) + \widetilde{\mathcal{C}(X)} \subset \mathfrak{M}_0 = cl_{BMO}\, \mathcal{C}(X) \subset \mathfrak{M}.$

PROOF. The first inclusion will follow if we can show that every function in $\mathcal{C}(X)$ is in $\mathfrak{M}_0$. Let $v \in \mathcal{C}(X)$ with $\int v\, d\mu = 0$, so also $\int \tilde{v}\, d\mu = 0$. Since $(vT)\tilde{\,} = \tilde{v}T$ and $\|\tilde{v}\|_{BMO} \leq c\|v\|_\infty$ (see [3]), $\|M_n \tilde{v}\|_{BMO} \leq c\|M_n v\|_\infty \to 0$. Also,

$$\tilde{v} - \tilde{v}T = \sum_{k \neq 0,1} vT^k \left(\frac{1}{k} - \frac{1}{k-1} \right) + vT + v \overset{\cdot}{\in} \mathcal{C}(X),$$

since the series is absolutely and uniformly convergent.

To show that $cl_{BMO}\, \mathcal{C}(X) \subset \mathfrak{M}_0$, note that if $f_k \in \mathcal{C}(X)$ and $f_k \to f$ in BMO, then already from the case $m = 2$ in the definition of $\|f - f_k\|_{BMO}$ we must have

$$0 = \lim_k \tfrac{1}{2} \|(f - fT) - (f_n - f_n T)\|_\infty.$$

Therefore $f - fT \overset{\cdot}{\in} \mathcal{C}(X)$. Since $\|M_n f - \int f\, d\mu\|_\infty \to 0$ for $f \in \mathcal{C}(X)$, it follows readily that $\|M_n f - \int f\, d\mu\|_{BMO} \to 0$ for $f \in cl_{BMO}\, \mathcal{C}(X)$.

Finally, if $f \in \mathfrak{M}_0$, then $f - fT, fT - fT^2, \ldots, fT^{n-2} - fT^{n-1} \dot{\in} \mathcal{C}(X)$, and summing and dividing by n gives $f - M_n f \dot{\in} \mathcal{C}(X)$. Since $f - M_n f \to f - \int f \, d\mu$ in BMO, we must have $f \in \mathrm{cl}_{\mathrm{BMO}} \mathcal{C}(X)$.

The space $\mathfrak{M}_0$ depends on the choice of the Jewett-Krieger representative of T, while $\mathfrak{M}$ does not. It would be interesting to determine the extent to which $\mathfrak{M}_0$ can vary with the choice of representative and to establish different characterizations of $\mathfrak{M}$. Also, it is possible that $\mathcal{C}(X) + \widetilde{\mathcal{C}(X)} = \mathfrak{M}_0$ in general.

The ergodic versions of BMO and VMO thus appear to be interesting spaces, but, unlike the classical case where examples are readily available, it is not immediately evident whether or not every system $(X, \mathfrak{B}, \mu, T)$ has unbounded ergodic BMO and VMO functions. Indeed, only considering the case $m = 2$ in (1.2) above shows that any such function f must satisfy $f - f \circ T \in L^\infty$ (or $\mathcal{C}(X)$), $f \notin L^\infty$ (or $\mathcal{C}(X)$) and hence is likely to be rather complicated to describe. Because of the results of [2], it appears that if $f \in L^1$ and

$$f^*(x) = \sup_{n > 1} \left| (1/n) \sum_{k=0}^{n-1} f(T^k x) \right|,$$

then $\log f^* \in \mathrm{BMO}$. Here we show how to use Kakutani-Rokhlin towers to construct explictly unbounded ergodic BMO and VMO functions for each ergodic measure-preserving transformation T of a Lebesgue space (separable nonatomic probability space) $(X, \mathfrak{B}, \mu)$. We owe two of the key ideas behind the construction to earlier work of Anosov [1] and Conze [4], and we have also benefited from conversations with J. Brezin and B. Marcus.

THEOREM 2. *Let $T: X \to X$ be an ergodic measure-preserving transformation of a Lebesgue space $(X, \mathfrak{B}, \mu)$. Then T admits unbounded ergodic VMO (and hence BMO) functions.*
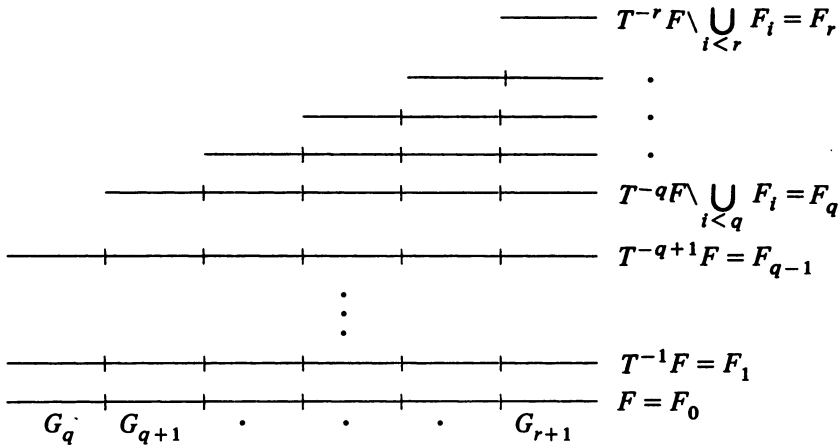
3. A topological Kakutani-Rokhlin Lemma. The remainder of the paper consists of a proof of Theorem 2. We assume, as we may be taking a finite-entropy factor if necessary and using the Jewett-Krieger Theorem, that (X, T) is a strictly ergodic subshift. A similar result to the following is found in [6].

LEMMA 1. *If (X, T) is a nonatomic strictly ergodic subsystem of the shift transformation on the space of sequences with entries from a finite alphabet, then for every integer $q > 0$ it is possible to find an open-closed subset E of X such that E, $TE, \ldots, T^{q-1}E$ are pairwise disjoint and $E, TE, \ldots, T^{2q-1}E$ cover X.*

PROOF. The proof of this statement is not much different from that of the well-known lemma of Rokhlin [14] and Kakutani [10]. First select a nonempty open-closed set F such that $F, T^{-1}F, \ldots, T^{-q+1}F$ are pairwise disjoint. Such a set can be constructed by fixing $x \in X$, choosing disjoint open-closed neighborhoods U_i of $T^{-i}x$ for $i = 0, \ldots, q - 1$, and letting $F = \bigcap_{i=0}^{q-1} T^i U_i$. Consider now the first-entry decomposition of X with respect to F; that is, for some smallest integer $r \geqslant q$, $F \cup T^{-1}F \cup \cdots \cup T^{-r}F = X$, and if we let

$$F_0 = F, \quad F_1 = T^{-1}F \setminus F_0, \ldots, F_r = T^{-r}F \setminus \bigcup_{i < r} F_i$$

and $G_k = \{x \in F \colon T^{-k}x \in F$ but $T^{-j}x \notin F$ for $j < k\}$ for $k = 1, 2, \ldots, r + 1$, then X is decomposed into open-closed subsets as in the following diagram:

$$
\begin{array}{c}
\rule{2cm}{0.4pt} \quad T^{-r}F\backslash \bigcup_{i<r} F_i = F_r \\[4pt]
\vdots \\
\rule{5cm}{0.4pt} \quad T^{-q}F\backslash \bigcup_{i<q} F_i = F_q \\[4pt]
\rule{6cm}{0.4pt} \quad T^{-q+1}F = F_{q-1} \\[4pt]
\vdots \\
\rule{6cm}{0.4pt} \quad T^{-1}F = F_1 \\[4pt]
\rule{6cm}{0.4pt} \quad F = F_0 \\[4pt]
G_q^{\cdot} \quad G_{q+1} \quad \cdot \quad \cdot \quad \cdot \quad G_{r+1}
\end{array}
$$

We now form the set E by including from each column of the above diagram the top-most set and, working downwards, every qth set, but not including any subsets of $F_0, F_1, \ldots,$ or F_{q-2}. More precisely,

$$
E = \bigcup_{i=0}^{r-q+1} \bigcup_{j=0}^{[i/q]} T^{jq}T^{-q-i+1}G_{q+i}.
$$

It is now clear that as we apply T, the various pieces of E move downwards through the diagram in such a way that $E, TE, \ldots, T^{q-1}E$ are pairwise disjoint and $E \cup TE \cup \cdots \cup T^{2q-1}E = X$,

4. Definition of f. The function $f \in \mathrm{BMO} \setminus L^\infty$ will be the sum of an infinite series, $f = \sum_{n=1}^\infty f_n$, where each f_n is a continuous function on X defined in terms of certain integers q_n and k_n. The sequences will be generated inductively so as to satisfy

$$
\sum_{n=1}^\infty 2^n \frac{k_{n+1} + 1}{q_{n+1}} < \infty, \tag{4.1}
$$

$$
\sum_{n=1}^\infty \frac{q_n}{k_{n+1}} < \infty, \tag{4.2}
$$

and

$$
\frac{1}{m} \sum_{j=0}^{m-1} f_n(T^j x) < 2\int_X f_n \, d\mu \quad \text{for all } m \geq q_{n+1}. \tag{4.3}
$$

Condition (4.1) will guarantee that $\sum_{n=1}^\infty \int f_n \, d\mu < \infty$ and hence the series for f converges a.e., and conditions (4.2) and (4.3) will be needed to verify (1.2).

To begin, using §3 choose a nonempty open-closed subset A_1 of X such that $A_1 \cap TA_1 = \varnothing$ and $A_1 \cup TA_1 \cup T^2A_1 \cup T^3A_1 = X$, let $q_1 = k_1 = 2$, and define $f_1 \equiv 1$ on X. Suppose that continuous nonnegative functions f_i, integers q_i and $k_i < q_i/2$, and open-closed sets A_i have been defined for $i = 1, \ldots, n$. For each

$E \subset X$ with $\mu(E) > 0$, denote by T_E the first-return map to E; that is, if $n_E(x) = \inf\{n > 0: T^n x \in E\}$, then

$$T_E x = T^{n_E(x)} x \quad \text{for } x \in E.$$

Let us abbreviate T_{A_i} by T_i. We now choose k_{n+1} consistently with (4.2)–for example,

$$k_{n+1} = 2^{n+1} q_n \tag{4.4}$$

will do–and then choose an even integer $q_{n+1} > q_n$ so as to satisfy both (4.1), say

$$q_{n+1} > 3^{n+1}(k_{n+1} + 1), \tag{4.5}$$

and (4.3). Such a choice of q_{n+1} is possible because T is strictly ergodic and f_n is continuous (see [12]). The integers k_{n+1} and q_{n+1} will then be used to define f_{n+1} as follows. Select an open-closed subset A_{n+1} of A_n such that $A_{n+1}, T_n A_{n+1}, \ldots, T_n^{q_{n+1}-1} A_{n+1}$ are pairwise disjoint and $A_{n+1}, T_n A_{n+1}, \ldots, T_n^{2q_{n+1}-1} A_{n+1}$ cover A_n, and define f_{n+1} on $A_{n+1} \cup T_n A_{n+1} \cup \cdots \cup T_n^{q_{n+1}-1} A_{n+1}$ to be the "spike function" which is 1 at the "center", $T_n^{q_{(n+1)/2}-1} A_{n+1} \cup T_n^{q_{(n+1)/2}} A_{n+1}$, and decreases in steps of $1/k_{n+1}$ towards the two "ends" A_{n+1} and $T_n^{q_{n+1}-1} A_{n+1}$. Since k_{n+1} is small compared to q_{n+1}, f_{n+1} takes the value 0 on most of the sets $T_n^j A_{n+1}$, $0 \leq j \leq q_{n+1} - 1$. In analytic terms, f_{n+1} is defined by

$$f_{n+1} \big| T_n^{q_{n+1}/2+j} A_{n+1} = f_{n+1} \big| T_n^{q_{n+1}/2-j-1} A_{n+1} \equiv 1 - \frac{j}{k_{n+1}} \quad \text{for } 0 \leq j \leq k_{n+1},$$

and

$$f_{n+1} \big| T_n^{q_{n+1}/2+j} A_{n+1} \equiv 0 \quad \text{if } |j| > k_{n+1} + 1.$$

The definition of f_{n+1} is propagated forward to the rest of X along the orbits: for each x let $f_{n+1}(x) = f_{n+1}(T^{-k}x)$, where $k = \inf\{i \geq 0: T^{-i}x \in \cup_{j=0}^{q_{n+1}-1} T_n^j A_{n+1}\}$. Then $f_{n+1}(x) = f_{n+1}(Tx)$ unless $Tx \in A_n$, in which case we still have

$$|f_{n+1}(x) - f_{n+1}(Tx)| \leq \frac{1}{k_{n+1}}. \tag{4.6}$$

In order to visualize this scheme of definition, note that f_{n+1} has essentially the same graph on A_n as on each of its copies $T_{n-1}A_n, T_{n-1}^2 A_n, \ldots, T_{n-1}^{q_n-1} A_n$. This graph is repeated in each copy of A_{n-1}, and so forth up to X itself. Also, f_{n+1} is constant on each element of a certain partition of X into finitely many open-closed subsets and therefore is continuous on X.

5. The integral of f_{n+1}. Note first that

$$\int_{A_n} f_{n+1} \, d\mu = 2 \sum_{j=1}^{k_{n+1}} \frac{j}{k_{n+1}} \mu(A_{n+1}) = (k_{n+1} + 1)\mu(A_{n+1}).$$

Since $\int_{T_{n-1}^j A_n} f_{n+1} \, d\mu = \int_{A_n} f_{n+1} \, d\mu$ for $j = 0, \ldots, q_n - 1$, and

$$\int_{T_{n-1}^{j-1} A_n \setminus \cup_{i<j+1} T_{n-1}^i A_n} f_{n+1} \, d\mu \leq \int_{T_{n-1}^j A_n \setminus \cup_{i<j} T_{n-1}^i A_n} f_{n+1} \, d\mu$$

for $j = q_n - 1, \ldots, 2q_n - 2$, it follows that

$$\int_{A_{n-1}} f_{n+1} \, d\mu \leqslant 2q_n \int_{A_n} f_{n+1} \, d\mu.$$

Similarly,

$$\int_{A_{n-2}} f_{n+1} \, d\mu \leqslant 2q_{n-1} \int_{A_{n-1}} f_{n+1} \, d\mu \leqslant 2^2 q_{n-1} q_n \int_{A_n} f_{n+1} \, d\mu;$$

after a suitable number of repetitions of this type of estimate, we see that

$$\int_X f_{n+1} \, d\mu \leqslant 2^n q_1 q_2 \cdots q_n \int_{A_n} f_{n+1} \, d\mu \leqslant 2^n q_1 q_2 \cdots q_n (k_{n+1} + 1) \mu(A_{n+1})$$

$$\leqslant 2^n \frac{k_{n+1} + 1}{q_{n+1}}.$$

By (4.1), $\sum_{n=1}^{\infty} \int_X f_n \, d\mu < \infty$, and this implies that $\sum_{n=1}^{\infty} f_n(x)$ converges a.e. Therefore $f = \sum_{n=1}^{\infty} f_n$ is well defined.

6. Proof that $f \in \mathrm{BMO} \setminus L^{\infty}$. From the construction of f it is clear that for each $n = 1, 2, \ldots$ there is a set of positive measure on which $f_1 \equiv f_2 \equiv \cdots \equiv f_n \equiv 1$. Then clearly $f = \sum_{n=1}^{\infty} f_n \notin L^{\infty}$.

In order to verify (1.2), fix $m = 1, 2, \ldots$ and let

$$\theta_n(x) = \frac{1}{m} \sum_{k=0}^{m-1} \left| f_n(T^k x) - \frac{1}{m} \sum_{j=0}^{m-1} f_n(T^j x) \right| \quad \text{for } n = 1, 2, \ldots.$$

Then

$$\frac{1}{m} \sum_{k=0}^{m-1} \left| f(T^k x) - \frac{1}{m} \sum_{j=0}^{m-1} f(T^j x) \right| \leqslant \sum_{n=1}^{\infty} \theta_n(x).$$

We split up the latter sum into three parts, consisting of the n's for which m looks large, small, and moderate, respectively:

$$\sum_{n=1}^{\infty} \theta_n(x) = \sum_{m > q_{n+1}} \theta_n(x) + \sum_{m < q_{n-1}} \theta_n(x) + \sum_{q_{n-1} \leqslant m < q_{n+1}} \theta_n(x).$$

In order to estimate the first term, observe that for $m > q_{n+1}$ we have

$$\theta_n(x) \leqslant 2 \frac{1}{m} \sum_{k=0}^{m-1} f_n(T^k x) < 4 \int_X f_n \, d\mu$$

by (4.3); therefore

$$\sum_{m > q_{n+1}} \theta_n(x) < 4 \sum_{n=1}^{\infty} \int_X f_n \, d\mu < \infty.$$

For the second term, we use (4.6), which implies that $|f_n(T^k x) - f_n(T^j x)| \leqslant q_{n-1}/k_n$ for $j, k = 0, 1, \ldots, m - 1$ provided that $m < q_{n-1}$. This yields

$$\sum_{m < q_{n-1}} \theta_n(x) < \sum_{m < q_{n-1}} \frac{1}{m} \sum_{k=0}^{m-1} \frac{1}{m} \sum_{j=0}^{m-1} |f_n(T^k x) - f_n(T^j x)| < \sum_{n=2}^{\infty} \frac{q_{n-1}}{k_n} < \infty$$

by (4.2). Finally, the sum comprising the third term, $\Sigma_{q_{n-1} \leqslant m < q_{n+1}} \theta_n$, consists of exactly two terms, since $\{q_n\}$ is increasing. Each of these terms is at most $2\|f_n\|_\infty < 2$, so the third term is bounded by 4. Combining these results shows that

$$\frac{1}{m} \sum_{k=0}^{m-1} \left| f(T^k x) - \frac{1}{m} \sum_{j=0}^{m-1} f(T^j x) \right| < 4 \sum_{n=1}^{\infty} \int_X f_n \, d\mu + \sum_{n=1}^{\infty} \frac{q_{n-1}}{k_n} + 4,$$

a finite constant independent of m. This proves that $f \in$ BMO.

7. An unbounded ergodic VMO function. If $g = \Sigma_{n=1}^{\infty} f_n/n$, then as before $g \notin L^\infty$. However, since

$$\lim_{n \to \infty} \left\| \sum_{k > n} \frac{f_k}{k} \right\|_{\text{BMO}} = 0,$$

g is in the BMO closure of $\mathcal{C}(X)$.

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