

ON THE INTEGRABILITY OF THE MAXIMAL ERGODIC FUNCTION

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ABSTRACT. Let $G = \mathbb{R}^d$ or \mathbb{Z}^d and consider an ergodic measure-preserving action of G on a probability space (X, \mathfrak{A}, P) , let $f \in L^1(X, P)$ and Mf be its maximal ergodic function. Our purpose is to prove the converse of the following theorem of N. Wiener: if $|f| \log^+ |f|$ is integrable then Mf is integrable. For the particular case $G = \mathbb{Z}$ this result was already obtained by D. Ornstein whose proof is based on induced transformations and seems to be specific to \mathbb{Z} , our proof is based on a result of E. M. Stein on the Hardy-Littlewood maximal function on \mathbb{R}^d and its analogue on \mathbb{Z}^d .

Introduction. Our main result is the “only if” part of the following theorem (see the notations below).

THEOREM. Let $G = \mathbb{R}^d$ or \mathbb{Z}^d with $d > 1$. Consider an ergodic measure-preserving action of G on a probability space (X, \mathfrak{A}, P) , and let f be a positive integrable function on (X, \mathfrak{A}, P) . The maximal function Mf is integrable if and only if $f \log^+ f$ is integrable.

The “if” part of the theorem is classical and was proved by N. Wiener (cf. [8], [2]), the “only if” part was proved for the particular case $G = \mathbb{Z}$ by D. Ornstein (cf. [6], [5]), with a proof based on induced transformations. Our proof is quite different and is based on a result of E. M. Stein (cf. [7]) on the Hardy-Littlewood maximal function. Similar results on certain classes of martingales were proved by D. L. Burkholder (cf. [1]) and R. F. Gundy (cf. [4]).

Let $G = \mathbb{R}^d$ or \mathbb{Z}^d with $d > 1$, let (X, \mathfrak{A}, P) be a probability space. We assume that G acts measurably by measure-preserving transformations on (X, \mathfrak{A}, P) : we denote this action by $G \times X \ni (g, x) \rightarrow gx \in X$ and write $\tau_g x = gx$.

Let μ be the Lebesgue measure on G , if V is a measurable subset of G , we write $|V| = \mu(V)$, and $\mu(dg) = dg$. If $G = \mathbb{R}^d$, we denote by V_r the ball of radius r centered at 0. If f is a measurable function on \mathbb{R}^d , we define the *Hardy-Littlewood maximal function* Mf of f by

$$Mf(g) = \sup_{r>0} \frac{1}{|V_r|} \int_{V_r} |f(g+h)| dh.$$

Now if $f \in L^1(X, \mathfrak{A}, P)$, then the function $G \ni g \rightarrow f_x(g) = f(gx)$ is μ -integrable on compact subsets of G for almost every x . We define

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$$Mf_x(g) = \sup_{r>0} \frac{1}{|V_r|} \int_{V_r} |f_x(g+h)| dh.$$

We write also $Mf(gx) = Mf_x(g)$, $x \in X, g \in \mathbb{R}^d$.

We denote by $\log^+|f|$ the positive part of $\log|f|$: $\log^+|f| = \max(\log|f|, 0)$.

When $G = \mathbb{Z}^d$ we note $V_n = \{-n, -n+1, \dots, n\}^d$ for $n > 1$, we define similarly Mf for $f \in L^1(\mathbb{Z}^d)$ and $Mf_x(g)$, $x \in X, g \in \mathbb{Z}^d$ for $f \in L^1(X, \mathfrak{A}, P)$.

The function $Mf(x)$ is called the *maximal ergodic function* of f . We recall the following result of E. M. Stein:

LEMMA 1 (cf. [7, p. 306]). *Let $f \in L^1(\mathbb{R}^d)$, then for $\lambda > 0$, we have*

$$|\{g \in \mathbb{R}^d | M|f|(g) > \lambda\}| > \frac{1}{c3^d\lambda} \int_{\{|f|>c\lambda\}} |f(g)| dg,$$

where c is a constant depending only on the dimension d .

Next, we prove the Calderón-Zygmund lemma (cf. [3, p. 91]) and Lemma 1 for \mathbb{Z}^d :

We call *quasi-cube* of \mathbb{Z}^d any subset of \mathbb{Z}^d of the form $I_1 \times \dots \times I_d$ where I_i is an interval of \mathbb{Z} , $i = 1, \dots, d$, satisfying

$$\sup_{1 \leq i, j \leq d} ||I_i| - |I_j|| < 1.$$

The number $\sup |I_i|$ is called the length of the quasi-cube. Any quasi-cube $1 < i < d$ with length > 3 can be divided into 2^d disjoint quasi-cubes (we divide each I_i into two intervals I_i^1 and I_i^2 with $||I_i^2| - |I_i^1|| < 1$ and form the new quasi-cubes as the products of these intervals) and any quasi-cube with length 2 is the disjoint union of at most 2^d one-point cubes.

LEMMA 2. *Let Q be a quasi-cube, $\lambda > 0$, f a positive function on Q such that*

$$\frac{1}{|Q|} \sum_{q \in Q} f(q) < \lambda.$$

There exist disjoint quasi-cubes Q_1, \dots, Q_n of Q such that if $q \notin \cup_{k=1}^n Q_k$ we have $f(q) < \lambda$ and $\lambda < |Q_k|^{-1} \sum_{q \in Q_k} f(q) < 3^d \lambda$, $k = 1, \dots, n$.

PROOF. Let P_1, \dots, P_n be a partition of Q into disjoint quasi-cubes as above (with $n_1 < 2^d$).

For each $P_j, j < n_1$ we have two alternatives: Either

(α) $|P_j|^{-1} \sum_{q \in P_j} f(q) < \lambda$; we have two possibilities:

if $|P_j| > 1$ we continue to subdivide P_j , if $|P_j| = 1, P_j = \{q\}$ we have $f(q) < \lambda$; or

(β) $|P_j|^{-1} \sum_{q \in P_j} f(q) > \lambda$; in this case we do have $|P_j|^{-1} \sum_{q \in P_j} f(q) < 3^d \lambda$ since

$$|Q|\lambda > \sum_{q \in Q} f(q) > \sum_{q \in P_j} f(q) \quad \text{and} \quad |P_j| > \frac{|Q|}{3^d}.$$

We keep then P_j as one of the Q_k 's.

An iteration of these arguments proves the lemma.

We deduce from this lemma the following version of the Calderón-Zygmund lemma (cf. [3, p. 91]) for \mathbb{Z}^d .

LEMMA 3. Let f be a positive integrable function on \mathbb{Z}^d , $\lambda > 0$. There exist disjoint quasi-cubes Q_1, \dots, Q_n such that if $g \notin \cup_{k=1}^n Q_k$ we have $f(g) < \lambda$ and

$$\lambda < \frac{1}{|Q_k|} \sum_{g \in Q_k} f(g) < 3^d \lambda, \quad k = 1, \dots, n.$$

PROOF. Let Q be a cube of \mathbb{Z}^d such that

$$f(g) < \lambda, \quad \text{if } g \notin Q \quad \text{and} \quad \frac{1}{|Q|} \sum_{q \in Q} f(q) < \lambda,$$

we can then apply the above lemma to Q .

As a consequence, we obtain a version of Lemma 1 of E. M. Stein for \mathbb{Z}^d .

LEMMA 4. Let f be a positive integrable function on \mathbb{Z}^d , let

$$(Mf)(g) = \sup_{n > 0} \frac{1}{|V_n|} \int_{V_n} f(g+h) dh,$$

where $V_n = \{-n, \dots, n\}^d$. For every $\lambda > 0$ we have

$$|\{Mf > \lambda\}| > \frac{1}{3^d c \lambda} \int_{\{f > c\lambda\}} f(g) dg$$

where c is a constant depending only on the dimension d (and is not equal to that of Lemma 1).

PROOF. We follow the proof of E. M. Stein:

By Lemma 3, there exist disjoint quasi-cubes Q_1, \dots, Q_n of \mathbb{Z}^d such that

$$f(g) < \lambda \quad \text{if } g \notin Q_1 \cup \dots \cup Q_n,$$

$$\lambda < \frac{1}{|Q_k|} \int_{Q_k} f(g) dg < 3^d \lambda, \quad k = 1, \dots, n.$$

It follows that if $g \in Q_1 \cup \dots \cup Q_n$ we have $Mf(g) > \lambda/c$, where $c = 3^d$. We deduce that $\{g \in \mathbb{Z}^d | Mf(g) > \lambda/c\} \supset \cup_{k=1}^n Q_k$ and therefore

$$\begin{aligned} \left| \left\{ g \in \mathbb{Z}^d | Mf(g) > \frac{\lambda}{c} \right\} \right| &> \sum_{k=1}^n |Q_k| \\ &> \frac{1}{3^d \lambda} \int_{\cup_{k=1}^n Q_k} f(g) dg > \frac{1}{3^d \lambda} \int_{\{f > \lambda\}} f(g) dg. \end{aligned}$$

The lemma follows if we replace λ by $c\lambda$.

LEMMA 5. Assume that the action of G on (X, \mathfrak{A}, P) is ergodic.

Let f be a positive integrable function on (X, P) . Then for every $\lambda > \|f\|_1$, we have the inequality

$$P(x \in X | Mf(x) > \lambda) > \frac{1}{\lambda 6^d c} \int_{\{f > c\lambda\}} f(x) dP(x). \tag{1}$$

PROOF. (a) By the pointwise ergodic theorem (cf. [8]) we have

$$\frac{1}{|V_n|} \int_{V_n} f(gx) dg \xrightarrow{n \rightarrow \infty} E(f) \quad P\text{-a.e.}$$

Let $\lambda > E(f)$, $\epsilon > 0$, using Egorov's theorem, we can find a measurable subset $X_\epsilon \subset X$, $P(X_\epsilon) > 1 - \epsilon$ and an integer n such that

$$\frac{1}{|V_n|} \int_{V_n} f(gx) dg < \lambda, \quad \forall x \in X_\epsilon. \tag{2}$$

Let $A = \{(g, x) \in (V_n - V_n) \times X | Mf_x(g) > \lambda\}$, where $V_n - V_n = \{g - g' | g \in V_n, g' \in V_n\}$,

$$A_g = A \cap (\{g\} \times X) = \{x \in X | Mf_x(g) > \lambda\}, \quad \forall g \in V_n - V_n,$$

$$A_0 = \{x \in X | Mf(x) > \lambda\},$$

$$A_x = A \cap (G \times \{x\}) = \{g \in (V_n - V_n) | Mf_x(g) > \lambda\}.$$

By the measure-preserving property, we have

$$P(A_g) = P(\tau_g^{-1}A_0) = P(A_0), \quad \forall g \in V_n - V_n. \tag{3}$$

Therefore by the Fubini theorem, we obtain

$$\begin{aligned} \int_{V_n - V_n} P(A_g) dg &= (\mu \otimes P)(A), \\ |(V_n - V_n)|P(A_0) &= (\mu \otimes P)(A), \\ |(V_n - V_n)|P(A_0) &= \int_X |A_x| dP(x). \end{aligned} \tag{4}$$

(b) Now let $f_x^{V_n}(g) = 1_{V_n}(g)f(\tau_g x)$, $(g, x) \in G \times X$, and let

$$A^{V_n} = \{(g, x) \in G \times X | Mf_x^{V_n}(g) > \lambda\},$$

$$A_x^{V_n} = A^{V_n} \cap (G \times \{x\}),$$

$$A_x^{V_n} = \{g \in G | Mf_x^{V_n}(g) > \lambda\}.$$

We remark that

$$Mf_x(g) > Mf_x^{V_n}(g), \quad x \in X, g \in G. \tag{5}$$

(c) Let $x \in X_\epsilon$, we shall show that

$$A_x^{V_n} \subset A_x, \tag{6}$$

or equivalently

$$\{g \in G | Mf_x^{V_n}(g) > \lambda\} \subset \{g \in (V_n - V_n) | Mf_x(g) > \lambda\}. \tag{7}$$

As $Mf_x(g) > Mf_x^{V_n}(g)$, $\forall (g, x) \in G \times X$, it suffices to show that

$$\{g \in G | Mf_x^{V_n}(g) > \lambda\} \subset (V_n - V_n),$$

but this relation follows from (2) and the remark that if $g \notin (V_n - V_n)$, we have

$$\frac{1}{|V_s|} \int_{V_s} f_x^{V_n}(h + g) dh \begin{cases} = 0 & \text{if } s < n, \\ < \frac{1}{|V_n|} \int_{V_n} f(hx) dh & \text{if } s > n. \end{cases}$$

(d) Let

$$\eta_\epsilon = \sup_{\substack{E \in \mathfrak{M} \\ P(E) < \epsilon}} \int_E f(x) dP(x), \quad \epsilon > 0.$$

As f is integrable, we have

$$\eta_\varepsilon \rightarrow 0, \text{ when } \varepsilon \rightarrow 0. \tag{8}$$

(e) Consider the relation (3):

$$|(V_n - V_n)|P(A_0) = \int_{X \setminus X_\varepsilon} |A_x| dP(x) + \int_{X_\varepsilon} |A_x| dP(x).$$

Combining with the relation (6), we have

$$\begin{aligned} |(V_n - V_n)|P(A_0) &> \int_{X_\varepsilon} |A_x^{V_n}| dP(x), \\ |(V_n - V_n)|P(A_0) &> \int_{X_\varepsilon} dP(x) \left(\frac{1}{\lambda 3^d c} \int_{\{g \in G | f_x^{V_n}(g) > c\lambda\}} f_x^{V_n}(g) dg \right) \end{aligned}$$

(by Lemma 1 for $G = \mathbb{R}^d$ and Lemma 4 for $G = \mathbb{Z}^d$).

$$\begin{aligned} |(V_n - V_n)|P(A_0) &> \frac{1}{\lambda 3^d c} \int_X dP(x) \int_{\{g \in G | f_x^{V_n}(g) > c\lambda\}} f_x^{V_n}(g) dg \\ &\quad - \frac{1}{\lambda 3^d c} \int_{X \setminus X_\varepsilon} dP(x) \int_{\{g \in G | f_x^{V_n}(g) > c\lambda\}} f_x^{V_n}(g) dg. \end{aligned} \tag{9}$$

(f) But

$$\begin{aligned} \int_{X \setminus X_\varepsilon} dP(x) \int_{\{g \in G | f_x^{V_n}(g) > c\lambda\}} f_x^{V_n}(g) dg &< \int_{X \setminus X_\varepsilon} dP(x) \int_{V_n} f(gx) dg \\ &< \int_{V_n} dg \int_{X \setminus X_\varepsilon} f(gx) dP(x) \\ &< \int_{V_n} dg \int_X 1_{\tau_\varepsilon^{-1}(X - X_\varepsilon)}(x) f(x) dP(x). \\ \int_{X \setminus X_\varepsilon} dP(x) \int_{\{g \in G | f_x^{V_n}(g) > c\lambda\}} f_x^{V_n}(g) dg &< |V_n| \eta_\varepsilon \end{aligned} \tag{10}$$

(since $P(\tau_\varepsilon^{-1}(X \setminus X_\varepsilon)) = P(X \setminus X_\varepsilon) < \varepsilon$).

(g) Consider now

$$\begin{aligned} \int_X dP(x) \int_{\{g \in G | f_x^{V_n}(g) > c\lambda\}} f_x^{V_n}(g) dg \\ = \int_{V_n} dg \int_X 1_{\{x \in X | 1_{V_n}(g)(gx) > c\lambda\}} f(gx) dP(x), \end{aligned}$$

if $g \notin V_n$, we have

$$\int_X 1_{\{x \in X | 1_{V_n}(g)(gx) > c\lambda\}} f(gx) dP(x) = 0,$$

if $g \in V_n$, we have

$$\begin{aligned} \int_X 1_{\{x \in X | 1_{V_n}(g)(gx) > c\lambda\}} f(gx) dP(x) &= \int_X 1_{\{x \in X | f(x) > c\lambda\}} f(x) dP(x) \\ &= \int_{\{f > c\lambda\}} f(x) dP(x). \end{aligned}$$

Therefore

$$\int_X dP(x) \int_{\{g \in G | f_n^{V_n}(g) > c\lambda\}} f_x^{V_n}(g) dg = |V_n| \int_{\{f > c\lambda\}} f(x) dP(x). \quad (11)$$

(h) Combining (9), (10) and (11) we have

$$|V_n - V_n| P(A_0) > \frac{|V_n|}{\lambda 3^d c} \int_{\{f < c\lambda\}} f(x) dP(x) - \frac{|V_n|}{\lambda 3^d c} \eta_\epsilon. \quad (12)$$

As $|V_n| < |(V_n - V_n)| < 2^d |V_n|$, $A_0 = \{x \in X | Mf(x) > \lambda\}$ and $\lim_{\epsilon \rightarrow 0} \eta_\epsilon = 0$, we finally obtain

$$P(x \in X | Mf(x) > \lambda) > \frac{1}{\lambda 6^d c} \int_{\{f > c\lambda\}} f(x) dP(x).$$

The lemma is proved.

PROOF OF THE THEOREM. The theorem follows from the lemma by the following well-known argument:

$$\begin{aligned} \int_X Mf(x) dP(x) &= \int_0^\infty P(Mf > \lambda) d\lambda \\ &= \int_0^{\|f\|_1} P(Mf > \lambda) d\lambda + \int_{\|f\|_1}^\infty P(Mf > \lambda) d\lambda \\ &> C_1 + \int_{\|f\|_1}^\infty \frac{1}{6^d \lambda c} \int_{\{f > c\lambda\}} f(x) dP(x) d\lambda \\ &> C_2 + \frac{1}{6^d c} \int_X f(x) \int_1^{\max(f(x)/c, 1)} \frac{1}{\lambda} d\lambda \\ &> C_3 + \frac{1}{6^d c} \int_X f(x) \log^+ f(x) dP(x). \end{aligned}$$

We have proved the theorem.

REMARKS. (a) In the theorem, the hypothesis of the ergodicity of the system is important. It is easy to see that the conclusion of the theorem is true if and only if the system has a finite number of ergodic components: if the system possesses an infinite number of disjoint measurable invariant subsets we can easily construct an *invariant* positive integrable f such that $f \log^+ f$ is *not* integrable, but for such a function we have $Mf = f$.

(b) The proof of the theorem is not modified if we replace V_n by $\{0, 1, \dots, n-1\}^d$ in the case of \mathbf{Z}^d and V_r by $[0, r]^d$ in the case of \mathbf{R}^d .

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