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A CLASSIFICATION THEOREM FOR SKT-MODULES

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ABSTRACT. In this paper a class of Abelian groups which includes the torsion totally projective groups, S-groups, and balanced projectives is studied. It is shown that this class of groups has a complete set of invariants.

If p is a prime number then the ring of all rational numbers a/b with b relatively prime to p will be denoted by Z_p.

The category of Z_p-modules are those Abelian groups with the property that multiplication by a prime other than the prime p is an automorphism of the group. If M is a Z_p-module then (i) the torsion submodule of M is the maximal torsion subgroup of M; (ii) M is torsion-free if and only if the torsion submodule of M is (0); (iii) M is reduced if and only if M is a reduced group; and (iv) M is divisible if and only if M is a divisible group.

Z and Q will denote the groups of integers and rational numbers respectively. The group $\mathrm{Ext}(Q/Z_p, *)$ will be denoted by $c(*)$.

The limit ordinal λ is a limited ordinal cofinal with ω if there is a sequence of smaller ordinals β_i such that $\lambda = \sup \beta_i$. Otherwise λ is said to be not cofinal with ω.

Three families of invariants u, h, and k are defined for the group G as follows. The invariant $u(p^\alpha, G)$ (the Ulm invariant) for the prime p and ordinal α is the dimension of the Z/pZ-vector space $(p^\alpha G)[p]/(p^{\alpha+1}G)[p] \equiv U(p^\alpha, G)$. The invariant $h(p^\beta, G)$ for the prime p and ordinal β is the dimension of the Z/pZ-vector space $p^\beta G/(p^{\beta+1}G + T)$, where T is the maximal torsion subgroup of $p^\beta G$. The invariant $k(p^\lambda, G)$ for the prime p and limit ordinal λ such that λ is not cofinal with ω, is the dimension of the Z/pZ-vector space $K(p^\lambda, G) \equiv p^\lambda c(G/p^\lambda G)/p^{\lambda+1}c(G/p^\lambda G)$.

Warfield [2] and [3] showed that the family of invariants u and h classify the balanced projectives, and the family of invariants u and k classify the S-groups. Noting that the family of invariants h are 0 on the S-groups, and the family of invariants k are 0 on the balanced projectives, it was conjectured that the family of invariants u, h and k could be used to classify those groups which are the direct sum of S-groups and balanced projectives. These groups are called SKT-modules. It will be shown in [4] that SKT-modules are projective relative to a well-defined class of sequences. It then follows that the SKT-modules form a class of groups

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which contains the S-groups and balanced projectives, have a projective characterization, and a complete family of invariants.

For the limit ordinal λ, a Z_p-module M is a λ-elementary balanced projective if and only if $p^\lambda M \cong Z_p$ and $M/p^\lambda M$ is a totally projective p-group. The torsion subgroup of M is called a λ-elementary S-group. A Z_p-module M is a balanced projective if and only if it is isomorphic to the direct sum of a totally projective p-group and λ-elementary balanced projectives for various limit ordinals λ. The torsion subgroup of a balanced projective is called an S-group.

Definition. A module M is an SKT-module if and only if there is a balanced projective K and an S-group S such that M is isomorphic to $K \oplus S$.

Lemma. *If K is a balanced projective and λ is a limit ordinal not cofinal with ω, then $k(p^\lambda, K) = 0$.*

Proof. Write

$$K \cong T \oplus \Big(\bigoplus_{\beta \in \Gamma} M_\beta \Big)$$

where T is a totally projective p-group and M_β is the direct sum of β-elementary balanced projectives. Let λ be a limit ordinal such that λ is not cofinal with ω. If $K \cong A_\lambda \oplus B_\lambda$ where A_λ is the direct sum of T and M_β where $\beta > \lambda$, and B_λ is the direct sum of all M_β where $\beta < \lambda$, then $K/p^\lambda K \cong A_\lambda/p^\lambda A_\lambda \oplus B_\lambda$. Since $A_\lambda/p^\lambda A_\lambda$ is a totally projective p-group, $p^\lambda c(A_\lambda/p^\lambda A_\lambda) = 0$ by [**1**, 3.10]. If S is the maximal torsion subgroup of B_λ then S is the direct sum of p-groups of length less than λ and $0 = p^\lambda c(S) = p^\lambda c(B_\lambda)$ by [**1**, 3.10]. It has therefore been shown that $p^\lambda c(K/p^\lambda K) = 0$ and $k(p^\lambda, K) = 0$.

Theorem. *If A and B are SKT-modules such that $u(p^\alpha, A) = u(p^\alpha, B)$, $h(p^\beta, A) = h(p^\beta, B)$, and $k(p^\lambda, A) = k(p^\lambda, B)$ for all ordinals α, β and λ such that β and λ are limit ordinals and λ is not cofinal with ω, then A is isomorphic to B.*

Proof. Since A and B are SKT-modules, $A \cong K \oplus S$ and $B \cong \underline{K} \oplus \underline{S}$ where K and $\underline{K}$ are balanced projectives, and S and $\underline{S}$ are S-groups. It can be assumed that $S \cong \bigoplus_{\lambda \in \Lambda} S_\lambda$ and $\underline{S} \cong \bigoplus_{\lambda \in \Delta} \underline{S}_\lambda$ where S_λ and $\underline{S}_\lambda$ are the direct sum of λ-elementary S-groups, and if $\lambda \in \Lambda$ or $\lambda \in \Delta$ then λ is not cofinal with ω. Since $k(p^\lambda, A) = k(p^\lambda, S) = k(p^\lambda, B) = k(p^\lambda, \underline{S})$, $\Lambda = \Delta \equiv \nabla$. For each $\lambda \in \nabla$, let S_λ and $\underline{S}_\lambda$ be the torsion subgroups of the balanced projectives N_λ and $\underline{N}_\lambda$ respectively. Furthermore, it will be assumed that N_λ and $\underline{N}_\lambda$ are direct sums of λ-elementary balanced projectives. Note that for each $\lambda \in \nabla$, $p^\lambda N_\lambda \cong p^\lambda \underline{N}_\lambda$ because $h(p^\lambda, N_\lambda) = k(p^\lambda, S_\lambda) = k(p^\lambda, A) = k(p^\lambda, B) = k(p^\lambda, \underline{S}_\lambda) = h(p^\lambda, \underline{N}_\lambda)$, [**3**, 2.3]. Let $M \cong K \oplus (\bigoplus_{\lambda \in \nabla} N_\lambda)$ and $\underline{M} \cong \underline{K} \oplus (\bigoplus_{\lambda \in \nabla} \underline{N}_\lambda)$. Since K and $\underline{K}$ are balanced projectives, then $K \cong T \oplus (\bigoplus_{\beta \in \Gamma} K_\beta)$ and $\underline{K} \cong \underline{T} \oplus (\bigoplus_{\beta \in \psi} \underline{K}_\beta)$, where T and $\underline{T}$ are totally projective p-groups, and K_β and $\underline{K}_\beta$ are the direct sum of β-elementary balanced projectives. Since $h(p^\beta, K) = h(p^\beta, A) = h(p^\beta, B) = h(p^\beta, \underline{K})$, $\Gamma = \psi \equiv \Omega$, furthermore, if $\beta \in \Omega$ then $p^\beta K_\beta \cong p^\beta \underline{K}_\beta$.

Let

$$N = \Big(\bigoplus_{\beta\in\Omega} p^\beta K_\beta\Big) \oplus \Big(\bigoplus_{\lambda\in\nabla} p^\lambda N_\lambda\Big) \quad \text{and} \quad \underline{N} = \Big(\bigoplus_{\beta\in\Omega} p^\beta \underline{K}_\beta\Big) \oplus \Big(\bigoplus_{\lambda\in\nabla} p^\lambda \underline{N}_\lambda\Big).$$

Then there is a height preserving isomorphism ϕ: $N \to \underline{N}$ such that ϕ takes $p^\beta K_\beta$ onto $p^\beta \underline{K}_\beta$ and $p^\lambda N_\lambda$ onto $p^\lambda \underline{N}_\lambda$ for each $\beta \in \Omega$ and $\lambda \in \nabla$. Note that N is a p-nice submodule of M and $\underline{N}$ is a p-nice submodule of $\underline{M}$. In addition, M/N and $\underline{M}/\underline{N}$ are totally projective p-groups. For each α, $U(p^\alpha, M)/I(p^\alpha, N) \cong U(p^\alpha, \underline{M})/I(p^\alpha, \underline{N})$ since M and $\underline{M}$ have the same Ulm invariants and $I(p^\alpha, N) = I(p^\alpha, \underline{N}) = 0$. The map ϕ extends to an isomorphism ϕ^* of M onto $\underline{M}$ [3, 1.2]. The module A can be identified with the submodule consisting of all elements g in M such that for some integer n, $p^n g$ is in the submodule $\bigoplus_{\beta\in\Omega} p^\beta K_\beta$. The module B can be identified with a similar submodule of $\underline{M}$. Since the isomorphism ϕ^* takes the submodule $\bigoplus_{\beta\in\Omega} p^\beta K_\beta$ onto $\bigoplus_{\beta\in\Omega} p^\beta \underline{K}_\beta$, ϕ^* will take A onto B.

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