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# TAME MEASURES ON CERTAIN COMPACT SETS

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ABSTRACT. A finite complex Borel measure $\mu$ on a compact subset $X \subset \mathbf{C}^n$ is called *tame* if there exist finite measures $\sigma_1, \ldots, \sigma_n$ on $X$ with

$$\int_X \phi\, d\mu = \int_X \sum_1^n \frac{\partial \phi}{\partial \bar{z}_j}\, d\sigma_j$$

for every $\phi \in C_0^\infty(\mathbf{C}^n)$. We define $X_T = \{(z_1, z_2): |z_1|^2 + |z_2|^2 = 1 \text{ and } z_1 \in T\}$, where $T$ is a compact subset of $\{|z_1| < 1\}$ in $\mathbf{C}^1$. It is shown in this paper that tame measures form a weak-* dense subset of $R(X_T)^\perp$. It follows then, with the help of a theorem by Weinstock, that $R(X_T)$ is a local algebra.

Let $X$ be a compact set in $\mathbf{C}^n$. $C(X)$ is the algebra of all continuous functions on $X$. $R_0(X)$ is the algebra of all rational functions $P/Q$ on $\mathbf{C}^n$ with $P$, $Q$ polynomials and $Q \neq 0$ on $X$. $R(X)$ is the uniform closure of $R_0(X)$ in $C(X)$.

It is a well-known consequence of Cauchy-Green formula that if $\mu$ is a complex Borel measure with compact support $X \subset \mathbf{C}^1$, then

$$\int \phi\, d\mu = -\frac{1}{2\pi i}\int \frac{\partial \phi}{\partial \bar{z}}\left(\int \frac{1}{\zeta - z}\, d\mu(\zeta)\right) dz \wedge d\bar{z} \tag{1}$$

holds for every $\phi \in C_0^\infty(\mathbf{C})$. It follows that $\mu$ is an orthogonal measure for $R(X)$ iff

$$\hat{\mu} = \int \frac{1}{\zeta - z}\, d\mu(\zeta) \tag{2}$$

is supported on $X$, or, equivalently, the measure $\hat{\mu}(z) dz \wedge d\bar{z}$ is supported on $X$. This gives a description of orthogonal measures for $R(X)$ where $X \subset \mathbf{C}$. While no general description for measures on $X \subset \mathbf{C}^n$, $n > 1$, orthogonal to $R(X)$ is available, we introduce the following definition.

DEFINITION. Let $X$ be a compact set in $\mathbf{C}^n$. A finite complex Borel measure is *tame* if there exist finite measures $\sigma_1, \ldots, \sigma_n$ on $X$ with

$$\int_X \phi\, d\mu = \int_X \sum_1^n \frac{\partial \phi}{\partial \bar{z}_j}\, d\sigma_j \quad \text{for every } \phi \in C_0^\infty(\mathbf{C}^n). \tag{$*$}$$

(1) and (2) now imply that for $X \subset \mathbf{C}$ a measure $\mu$ on $X$ is orthogonal to $R(X)$ iff $\mu$ is tame.

Let $X \subset \mathbf{C}^n$ be fixed. Suppose that a tame measure $\mu$ exists on $X$ with $\mu \neq 0$. If $\phi \in R_0(X)$ then $\partial\phi/\partial\bar{z}_j \equiv 0$ on $X$ for all $j$. So by $(*)$ $\int_X \phi\, d\mu = 0$. It follows that $\mu \perp R(X)$ and hence $R(X) \neq C(X)$. Thus the existence of tame measures imply that $R(X) \neq C(X)$.

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In this paper, we restrict ourselves to subsets of $\partial B = \{(z_1, z_2): |z_1|^2 + |z_2|^2 = 1\}$ which have the form $X_T = \{(z_1, z_2) \in \partial B: z_1 \in T\}$ where $T$ is a compact subset of $\{|z_1| < 1\}$ in $\mathbf{C}$. We study orthogonal measures for $R(X_T)$ and problems related to these measures.

Basener [1] has constructed a compact subset $\tilde{X}_T$ (which has the above stated form) of $\partial B$ such that $\tilde{X}_T$ is rationally convex, yet $R(\tilde{X}_T) \neq C(\tilde{X}_T)$. In the following, we will construct an ample family of tame measures for $R(X_T)$, provided that $R(T) \neq C(T)$. In fact, they form a weak-* dense set of $R(X_T)^{\perp}$. This gives an alternative explanation why $R(\tilde{X}_T) \neq C(\tilde{X}_T)$. Moreover, the weak-* density along with a theorem of Weinstock [2] lead to the conclusion that $R(X_T)$ is a local algebra. I.e., if $\{U_\alpha\}$ is a finite open covering of $X_T$ and if $f \in C(X_T)$ is such that $f|_{X_T \cap \bar{U}_\alpha}$ is in $R(X_T \cap \bar{U}_\alpha)$, for all $\alpha$, then $f \in R(X_T)$. The main results may be stated as follows.

THEOREM 1. *Assume that $R(T) \neq C(T)$. Then the set of tame measures on $X_T$ is weak-* dense in the set of all orthogonal measures to $R(X_T)$ on $X_T$.*

THEOREM 2. *Let $\phi$ be a smooth function with $\partial\phi/\partial\bar{z}_i \equiv 0$ on $X_T$, $i = 1, 2$. Then $\phi \in R(X_T)$.*

THEOREM 3. *$R(X_T)$ is a local algebra.*

NOTATIONS.

$$B = \{(z_1, z_2): |z_1|^2 + |z_2|^2 < 1\},$$
$$\partial B = \{(z_1, z_2): |z_1|^2 + |z_2|^2 = 1\},$$
$$\Delta = \{z_1 \in \mathbf{C}: |z_1| < 1\},$$
$$X_T = \{(z_1, z_2) \in \partial B, z_1 \in T\} \quad \text{where } T \text{ is a compact subset of } \Delta,$$
$$\Gamma_{z_1} = \{(z_1, (1 - z_1\bar{z}_1)^{1/2}e^{i\theta}): -\pi \leqslant \theta < \pi\}.$$

Let $\phi$ be any smooth function in a neighborhood of $X_T$. Let $\tilde{\phi}$ denote the composite $\phi \circ p$ where $p$ is the map from $\{|z_1| \leqslant 1\} \times [-\pi, \pi]$ to $\partial B$ defined by $p(z_1, \theta) = (z_1, (1 - z_1\bar{z}_1)^{1/2}e^{i\theta})$. For each fixed $z_1 \in T$, $\tilde{\phi}$ has the following Fourier expansion on $\Gamma_{z_1}$:

$$\phi(z_1, z_2) = \tilde{\phi}(z_1, \theta) = \sum_{-\infty}^{\infty} \phi_n(z_1)e^{in\theta}, \qquad z_2 = (1 - z_1\bar{z}_1)^{1/2}e^{i\theta},$$

where

$$\phi_n(z_1) = \int_{-\pi}^{\pi} \tilde{\phi}(z_1, t)e^{-int}\,\frac{dt}{2\pi}$$

is the $n$th Fourier coefficient of $\tilde{\phi}(z_1, \theta)$.

It is well known that

(i) $\phi_n(z_1)$ is smooth in $z_1$,

(ii) if $n \neq 0$, $|\phi_n(z_1)| \leqslant M/n^3$ for all $z_1 \in T$ where $M$ is a constant depending on $\phi$.

THEOREM 1. *Assume that $R(T) \neq C(T)$. Then the set of tame measures on $X$ is weak-* dense in the set $R(X_T)^{\perp}$ of all measures on $X_T$ orthogonal to $R(X_T)$.*

PROOF. Let $\nu$ be a nonzero orthogonal measure for $R(T)$. Consider the linear functional which assigns to each $f$ in $C(X_T)$ the value

$$\int_T \left( \frac{1}{2\pi i} \int_{\Gamma_{z_1}} f(z_1, z_2) \frac{dz_2}{z_2} \right) d\nu(z_1).$$

Since $T$ is a compact subset of $\Delta$, we have $X_T \cap \{z_2 = 0\} = \phi$. Hence the above is well defined. It is easy to see that this linear functional is continuous, therefore it defines a measure $\mu$ on $X_T$, i.e.

$$\int f\, d\mu = \int \left( \frac{1}{2\pi i} \int_{\Gamma_{z_1}} f(z_1, z_2) \frac{dz_2}{z_2} \right) d\nu(z_1) \quad \text{for all } f \in C(X_T). \tag{**}$$

ASSERTION. $\mu$ is tame.
Let $\phi \in C_0^\infty(\mathbf{C}^2)$,

$$\begin{aligned}
\int_{X_T} \phi\, d\mu &= \int \left( \frac{1}{2\pi i} \int_{\Gamma_{z_1}} \phi(z_1, z_2) \frac{dz_2}{z_2} \right) d\nu(z_1) \\
&= \int_T \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} \tilde{\phi}(z_1, \theta)\, d\theta \right) d\nu(z_1) \\
&= \int_T \phi_0(z_1)\, d\nu(z_1) \\
&= \frac{-1}{2\pi i} \int \frac{\partial \phi_0}{\partial \bar{\zeta}_1} \hat{\nu}(\zeta_1)\, d\zeta_1 \wedge d\bar{\zeta}_1 \quad \text{by (1).}
\end{aligned}$$

Assume the following lemma which will be proved later.

LEMMA.

$$\frac{\partial \phi_0}{\partial \bar{\zeta}_1} = \left( \frac{\partial \phi}{\partial \bar{\zeta}_1} - \frac{\zeta_1}{\zeta_2} \frac{\partial \phi}{\partial \bar{\zeta}_2} \right)_0 \tag{3}$$

*the zeroth Fourier coefficient of* $\partial\phi/\partial\bar{\zeta}_1 - (\zeta_1/\zeta_2)(\partial\phi/\partial\bar{\zeta}_2)$.

We get that

$$\begin{aligned}
\int_{X_T} \phi\, d\mu &= \frac{-1}{2\pi i} \int \left( \frac{\partial \phi}{\partial \bar{\zeta}_1} - \frac{\zeta_1}{\zeta_2} \frac{\partial \phi}{\partial \bar{\zeta}_2} \right)_0 \hat{\nu}(\zeta_1)\, d\zeta_1 \wedge d\bar{\zeta}_1 \\
&= \frac{-1}{2\pi i} \int \left[ \frac{1}{2\pi i} \int_{\Gamma_{\zeta_1}} \left( \frac{\partial \phi}{\partial \bar{\zeta}_1} - \frac{\zeta_1}{\zeta_2} \frac{\partial \phi}{\partial \bar{\zeta}_2} \right) \frac{d\zeta_2}{\zeta_2} \right] \hat{\nu}(\zeta_1)\, d\zeta_1 \wedge d\bar{\zeta}_1.
\end{aligned}$$

Let $\sigma_1$ be the measure on $\partial B$ such that for $f$ in $C(\partial B)$,

$$\int f(z_1, z_2)\, d\sigma_1 \equiv \int \frac{1}{4\pi^2} \left( \int_{\Gamma_{z_1}} f(z_1, z_2) \frac{dz_2}{z_2} \right) \hat{\nu}(z_1)\, dz_1 \wedge d\bar{z}_1$$

and let $\sigma_2 = -(z_1/z_2)\sigma_1$.

Again, the above definitions are legitimate, for, $\nu \perp R(T)$ implies that $\hat{\nu}(z_1) = 0$ outside $T$. This also shows that $\sigma_1$, $\sigma_2$ are supported on $X_T$. To sum up, we have shown that for any $\phi \in C_0^\infty(\mathbf{C}^2)$,

$$\int \phi \, d\mu = \int \frac{\partial \phi}{\partial \bar{z}_1} \, d\sigma_1 + \int \frac{\partial \phi}{\partial \bar{z}_2} \, d\sigma_2$$

where $\sigma_i$'s are supported on $X_T$. Hence $\mu$ is tame. $\mu$ is not a zero measure because $\int f(z_1) \, d\mu = \int f(z_1) \, d\nu$ for all $f$ in $C(\mathbf{C}^1)$ and $\nu$ is nonzero by hypothesis.

We note that if $\mu$ is a tame measure on $X \subset \mathbf{C}^n$, then $f\mu$ is also tame for smooth function $f$ with $\partial f / \partial \bar{z}_i \equiv 0$ on $X$, $i = 1, \ldots, n$. In particular, if $\mu$ is as in (**), the measures $z_2^m \mu$, $m = \pm 1, \pm 2, \ldots$, are all tame. Let $S = \{z_2^m \mu$: there is a nonzero orthogonal measure $\nu$ for $R(T)$ such that $\mu$ is defined by (**), $m = 0, \pm 1, \pm 2, \ldots\}$.

We will show that

"If $f$ in $C(X_T)$ is such that $f$ is annihilated by all elements in $S$, then $f$ is in $R(X_T)$." (#)

Let $\sum_{-\infty}^{\infty} f_n(z_1) e^{in\theta}$ be the "formal" Fourier expansion for $\tilde{f}(z_1, \theta) = f \circ p(z_1, \theta) = f(z_1, z_2)$ on $\Gamma_{z_1}$. Let $\sigma_j(z_1, z_2) = \tilde{\sigma}_j(z_1, \theta)$ be the $j$th Cesàro mean for $\tilde{f}$. It is a straightforward generalization of Fourier series theory on the circle that $\sigma_j$ converges uniformly to $f$ on $X_T$. So, in order to show $f \in R(X_T)$, we need only to show $\sigma_j$'s in $R(X_T)$ for all $j$. Fix $z_2^m \mu$ in $S$,

$$\int \sigma_j z_2^m \, d\mu \equiv \int_T \left( \frac{1}{2\pi i} \int_{\Gamma_{z_1}} \sigma_j(z_1, z_2) z_2^m \frac{dz_2}{z_2} \right) d\nu(z_1)$$

$$= \begin{cases} 0, \; j < |m|, \\ \dfrac{j - |m| + 1}{j} \displaystyle\int_T f_{-m}(z_1)(1 - z_1 \bar{z}_1)^{m/2} \, d\nu(z_1), \; j > |m|. \end{cases}$$

As $j \to \infty$, $\int \sigma_j z_2^m \, d\mu \to \int f z_2^m \, d\mu = 0$ by hypothesis, while the right hand side approaches $\int_T f_{-m}(z_1)(1 - z_1\bar{z}_1)^{m/2} \, d\nu(z_1)$. So, we get $\int_T f_{-m}(z_1)(1 - z_1\bar{z}_1)^{m/2} \, d\nu(z_1) = 0$ for all $\nu$ in $R(T)^\perp$. Therefore, $f_{-m}(z_1)(1 - z_1\bar{z}_1)^{m/2} = h_{-m}(z_1)$ for some $h_{-m}(z_1)$ in $R(T)$. And

$$\begin{aligned} \sigma_j(z_1, z_2) &= \frac{1}{j} \sum_{n=0}^{j} \sum_{k=-n}^{n} f_k(z_1) e^{ik\theta} \\ &= \frac{1}{j} \sum_{n=0}^{j} \sum_{k=-n}^{n} h_k(z_1)(1 - z_1\bar{z}_1)^{k/2} e^{ik\theta} \\ &= \frac{1}{j} \sum_{n=0}^{j} \sum_{k=-n}^{n} h_k(z_1) z_2^k \end{aligned}$$

is in $R(X_T)$. So is $f$.

We can now assert that the linear span of $S$ is weak-* dense in $R(X_T)^\perp$. For, if not, then there exists $g$ in $C(X_T)$ such that $g$ annihilates $S$ as well as its linear span, yet $\int g\, d\tau \neq 0$ for some $\tau \in R(X_T)^{\perp}$ which is not in the span of $S$. By (#), $g$ is in $R(X_T)$. Hence $\int g\, d\tau = 0$, a contradiction. So the linear span of $S$ is weak-* dense in $R(X_T)^{\perp}$. Q.E.D.

PROOF OF LEMMA.

$$\frac{\partial \phi_0}{\partial \bar{\zeta}_1} = \frac{\partial}{\partial \bar{\zeta}_1}\left[\int_{-\pi}^{\pi} \tilde{\phi}(\zeta_1, t)\frac{dt}{2\pi}\right]$$

$$= \int_{-\pi}^{\pi}\left[\frac{\partial \phi}{\partial \bar{\zeta}_1} + \frac{\partial \phi}{\partial \zeta_2}\,\frac{-\zeta_1}{2(1-\zeta_1\bar{\zeta}_1)^{1/2}}e^{it} + \frac{\partial \phi}{\partial \bar{\zeta}_2}\,\frac{-\zeta_1}{2(1-\zeta_1\bar{\zeta}_1)^{1/2}}e^{-it}\right]\frac{dt}{2\pi}. \quad (4)$$

On the other hand,

$$\frac{\partial \tilde{\phi}}{\partial t} = \frac{\partial \phi}{\partial \zeta_2}\,\frac{\partial \zeta_2}{\partial t} + \frac{\partial \phi}{\partial \bar{\zeta}_2}\,\frac{\partial \bar{\zeta}_2}{\partial t} = i\frac{\partial \phi}{\partial \zeta_2}(1-\zeta_1\bar{\zeta}_1)^{1/2}e^{it} - i\frac{\partial \phi}{\partial \bar{\zeta}_2}(1-\zeta_1\bar{\zeta}_1)^{1/2}e^{-it}.$$

So,

$$\frac{\partial \phi}{\partial \zeta_2}e^{it} = \frac{1}{i}\frac{\partial \tilde{\phi}}{\partial t}\,\frac{1}{(1-z_1\bar{z}_1)^{1/2}} + \frac{\partial \phi}{\partial \bar{\zeta}_2}e^{-it}.$$

Substituting the above into (4), we get

$$\frac{\partial \phi_0}{\partial \bar{\zeta}_1} = \int_{-\pi}^{\pi}\left[\frac{\partial \phi}{\partial \bar{\zeta}_1} + \frac{1}{2i}\,\frac{-\zeta_1}{1-\zeta_1\bar{\zeta}_1}\,\frac{\partial \tilde{\phi}}{\partial t} + \frac{\partial \phi}{\partial \bar{\zeta}_2}\,\frac{-\zeta_1}{(1-\zeta_1\bar{\zeta}_1)^{1/2}}e^{-it}\right]\frac{dt}{2\pi}$$

$$= \int_{-\pi}^{\pi}\left(\frac{\partial \phi}{\partial \bar{\zeta}_1} + \frac{\partial \phi}{\partial \bar{\zeta}_2}\,\frac{-\zeta_1}{\zeta_2}\right)\frac{dt}{2\pi} = \left(\frac{\partial \phi}{\partial \bar{\zeta}_1} - \frac{\zeta_1}{\zeta_2}\,\frac{\partial \phi}{\partial \bar{\zeta}_2}\right)_0.$$

The term $\int(\partial\tilde{\phi}/\partial t)(dt/2\pi) = 0$, since, on $\Gamma_{\zeta_1}$, $\partial\tilde{\phi}/\partial t = \sum_{n=-\infty}^{\infty}(in)\phi_n e^{int}$ has no constant term. Q.E.D.

It is an immediate consequence of (#) that we have

THEOREM 2. *Let $\phi$ be a smooth function with $\partial\phi/\partial\bar{z}_i \equiv 0$ on $X_T$, $i = 1, 2$. Then $\phi \in R(X_T)$.*

PROOF. Since $\phi$ is annihilated by all elements of $S$ so by (#) $\phi \in R(X_T)$. Q.E.D.

Next, we state a theorem about tame measures in general which is derived from the proof of a theorem due to Weinstock [**2**, Theorem 1.4].

THEOREM (WEINSTOCK). *Let $X$ be a compact subset of $\mathbf{C}^n$. If $\mu$ is a tame measure on $X$ and $\{U_\alpha\}_1^N$ is a finite open covering of $X$, then there exist $\mu_\alpha$ orthogonal measures for $R(X \cap \overline{U}_\alpha)$, where each $\mu_\alpha$ has its support contained in $X \cap U_\alpha$, and $\mu = \sum_1^N \mu_\alpha$.*

PROOF. Let $\{\sigma_i\}_{i=1}^n$ be measures supported on $X$, such that $\mu = -\sum_{i=1}^n \partial\sigma_i/\partial\bar{z}_i$. Let $\{\phi_\alpha\}$ be a smooth partition of unity subordinate to $\{U_\alpha\}$ satisfying

(i) $0 \leqslant \phi_\alpha \leqslant 1$ and $\operatorname{supp} \phi_\alpha \subset U_\alpha$,

(ii) $\sum_1^N \phi_\alpha = 1$.

Then,

$$\begin{aligned}
\mu &= -\sum_{i=1}^n \frac{\partial}{\partial\bar{z}_i}\left(\left\{\sum_{\alpha=1}^N \phi_\alpha\right\}\sigma_i\right) \\
&= -\sum_{i=1}^n \sum_\alpha \frac{\partial\phi_\alpha}{\partial\bar{z}_i}\sigma_i - \sum_{i=1}^n \sum_\alpha \phi_\alpha \frac{\partial\sigma_i}{\partial\bar{z}_i} \\
&= -\sum_\alpha \sum_i \frac{\partial\phi_\alpha}{\partial\bar{z}_i}\sigma_i - \sum_\alpha \phi_\alpha \sum_i \frac{\partial\sigma_i}{\partial\bar{z}_i} \\
&= -\sum_\alpha \left\{-\sum \frac{\partial\phi_\alpha}{\partial\bar{z}_i}\sigma_i + \phi_\alpha\mu\right\} \\
&= \sum \mu_\alpha, \quad \text{where } \mu_\alpha = -\sum_{i=1}^n \frac{\partial\phi_\alpha}{\partial\bar{z}_i}\sigma_i + \phi_\alpha\mu.
\end{aligned}$$

To show that $\mu_\alpha \perp R(X \cap \bar{U}_\alpha)$, for any $g \in R_0(X \cap \bar{U}_\alpha)$,

$$\begin{aligned}
\int g\, d\mu_\alpha &= -\sum_{i=1}^n \int g \frac{\partial\phi_\alpha}{\partial\bar{z}_i}\, d\sigma_i + \int g\phi_\alpha\, d\mu \\
&= -\sum_{i=1}^n \int g \frac{\partial\phi_\alpha}{\partial\bar{z}_i}\, d\sigma_i + \sum_{i=1}^n \int \frac{\partial(g\phi_\alpha)}{\partial\bar{z}_i}\, d\sigma_i \\
&= -\sum_{i=1}^n \int g \frac{\partial\phi_\alpha}{\partial\bar{z}_i}\, d\sigma_i + \sum_{i=1}^n \int g \frac{\partial\phi_\alpha}{\partial\bar{z}_i}\, d\sigma_i \quad \text{since } \frac{\partial g}{\partial\bar{z}_i} = 0\ \forall i \\
&= 0.
\end{aligned}$$

So $\mu_\alpha$ annihilates $R_0(X \cap \bar{U}_\alpha)$, hence will annihilate its closure $R(X \cap \bar{U}_\alpha)$. Since $\operatorname{supp} \mu_\alpha \subset \operatorname{supp} \phi_\alpha \cap \operatorname{supp} \mu$; we have that $\operatorname{supp} \mu_\alpha \subset X \cap U_\alpha$ and the theorem is proved. Q.E.D.

With the help of (#) and the above theorem we can now assert that $R(X_T)$ is a local algebra.

THEOREM 3. *Let $\{U_\alpha\}$ be a finite open covering of $X_T$. Let $f \in C(X_T)$ be such that the restriction of $f$ to $X_T \cap \bar{U}_\alpha$ is in $R(X_T \cap \bar{U}_\alpha)$ for all $\alpha$. Then $f$ is in $R(X_T)$.*

PROOF. Let $S$ be as in Theorem 1. For any $\mu \in S$, $\mu$ is tame by Theorem 1. It follows from the above theorem that there exist $\mu_\alpha$'s such that $\operatorname{supp} \mu_\alpha \subset X_T \cap U_\alpha$, $\mu_\alpha \perp R(X_T \cap \bar{U}_\alpha)$ and $\sum \mu_\alpha = \mu$. Hence, by hypothesis,

$$\int_{X_T} f\, d\mu = \sum_\alpha \int_{X_T \cap \bar{U}_\alpha} f\, d\mu_\alpha = 0.$$

$f$ is annihilated by $S$, and $f$ is then in $R(X_T)$ by (#). Q.E.D.

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