

**THE SECOND COEFFICIENT OF
 UNIVALENT BIEBERBACH-EILENBERG FUNCTIONS
 NEAR THE IDENTITY**

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ABSTRACT. An asymptotic expansion is found for the maximum $|b_2|$, in terms of $\beta = 1 - |b_1|$, for functions $F(z) = b_1z + b_2z^2 + \dots$ which are in the class of univalent Bieberbach-Eilenberg functions and which are near the identity (b_1 near 1). The first two terms of this expansion are the same as the expansion of $|b_2|$ in terms of $1 - b_1$ for the functions which map the unit disc onto the interior of circles passing through ± 1 .

1. The class \mathcal{E} of univalent Bieberbach-Eilenberg functions consists of those functions $F(z) = b_1z + b_2z^2 + \dots$ which are analytic and univalent in the unit disc $U = \{z: |z| < 1\}$, satisfy $F(0) = 0$ and are such that $F(z) \cdot F(\zeta) \neq 1$, for any $z, \zeta \in U$ [1], [3]. Some of the interest in this class derives from the observation that if $f(z)$ is in \mathcal{S} , the usual class of normalized univalent functions, and if $1/q \notin f(U)$, then

$$F(z) = \frac{1 \mp (1 - qf(z))^{1/2}}{1 \pm (1 - qf(z))^{1/2}}$$

is in \mathcal{E} . Conversely, if $F \in \mathcal{E}$ then the functions

$$f(z) = \frac{F(z)}{F'(0)[1 \pm F(z)]^2} \tag{1.1}$$

are in \mathcal{S} [4]. It follows that $F(z) \in \mathcal{E}$ near the identity, $F(z) = z$, correspond in a natural way to $f(z)$ near the Koebe function, $f(z) = z/(1 + z)^2$.

It is interesting to observe that the transformation (1.1) is inherent in Bieberbach's paper [1]. Indeed the class \mathcal{E} appeared quite naturally in that paper when a transformation equivalent to (1.1) was applied as an aid in solving a specific extremal problem.

In [6], a method was developed for calculating the maximum of $|b_2|$ for a given $|b_1|$ in the class \mathcal{E} . Numerical calculations suggested that for $b_1 > 0$ and near 1, $1 - b_1 = o(|b_2|)$. See Figure 2 in [6]. These methods were extended in [5] to allow the computation of the entire b_1, b_2 coefficient body. There it was shown that for $b_1 > 0$ (as can be assumed) and near 1, b_2 lies in a roughly elliptical shaped region

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whose “semiminor axis” is $2b_1(1 - b_1)$ and whose “semimajor axis” is much larger so that the “ellipse” becomes very thin as $b_1 \rightarrow 1$.

An upper bound for the “semimajor axis” of this region is easily found. From (1.1) and the fact that the second coefficient of a function in \mathfrak{S} is bounded by 2, it follows that for a given b_1 , b_2 must lie in the lenticular region defined by $|b_2 \pm 2b_1^2| < 2b_1$. The half width of this region is $2b_1(1 - b_1)$, the same as the minimum dimension of the above mentioned “elliptical” region. The vertex of this lenticular region is at $2ib_1(1 - b_1^2)^{1/2}$, and this defines an upper bound for the maximum dimension of the b_2 region.

Numerical calculations indicate that for b_1 near 1 the functions in \mathfrak{E} which achieve the maximum $|b_2|$ appear to map U to nearly circular regions whose boundaries pass through ± 1 . See Figure 7(c) in [5]. This immediately raises the question of how closely the functions

$$F_r(z) = \frac{(1 - r^2)^{1/2}z}{(1 + irz)}, \quad 0 < r < 1, \tag{1.2}$$

which map U onto the interior of a circle passing through ± 1 come to achieving the maximum $|b_2|$. The functions (1.2), which are clearly in \mathfrak{E} , were shown by Jenkins [7] to maximize $|F(z)|$ with $|z| = r$ in the class \mathfrak{E} .

For a function $F_r(z)$ of (1.2), $b_1 = (1 - r^2)^{1/2}$ and $b_2 = ir(1 - r^2)^{1/2}$. Thus, for these functions $b_2 = ib_1(1 - b_1^2)^{1/2}$. This is exactly one half of the upper bound found above. Since the actual maximum must lie between these two, it is clear that $1 - b_1 = o(|b_2|)$ is true for the maximum $|b_2|$'s. The problem is, how much larger can the actual maximum be than the b_2 found from (1.2)?

In the remainder of this paper we obtain the first few terms of an asymptotic (or series) expansion of the maximum $|b_2|$ in terms of $\beta = 1 - b_1$ to show that the Jenkins function gives an extremely close approximation to this maximum. Indeed, the first two terms of the expansion coincide.

THEOREM. *If $F(z) = b_1z + b_2z^2 + \dots$ is such that $b_1 > 0$ and $|b_2|$ is a maximum for $F \in \mathfrak{E}$ with this b_1 , then*

$$b_2 = i2^{1/2}\beta^{1/2}\left(1 - \frac{5}{4}\beta + \frac{85}{3 \cdot 2^5}\beta^2 - \frac{55}{3 \cdot 2^7}\beta^3 + \dots\right) \tag{1.3}$$

for $\beta = 1 - b_1$ near 0. This may be compared with the value for b_2 given by the function $F_r(z)$ of (1.2),

$$b_2' = i2^{1/2}\beta^{1/2}\left(1 - \frac{5}{4}\beta + \frac{7}{2^5}\beta^2 + \frac{3}{2^7}\beta^3 + \dots\right) \tag{1.4}$$

and the minimum $|b_2|$ on the boundary of the set of all b_2 with b_1 fixed,

$$b_2'' = 2b_1(1 - b_1) = 2\beta(1 - \beta). \tag{1.5}$$

Observe that from (1.5) and (1.3) the maximum dimension of the “ellipse” is approximately the square root of the minimum dimension as $b_1 \rightarrow 1$.

Expansion (1.4) is obtained directly and can be left to the reader. The proof of (1.3) is outlined in the following sections. The calculations are straightforward but

rather tedious to do and check. It would be much more satisfactory to be able to obtain this expansion more directly, say from the Schiffer equation, but the writer sees no immediate way to do so that is not equivalent to what is done here. However, should such a method be devised, it can be checked against the expansion given in the theorem.

2. Some years ago, Jenkins [8] showed that the maximum $|b_2|$ in the class \mathcal{E} for a given b_1 was achieved by an (essentially unique) function in a one parameter family which he defined implicitly in terms of the mappings. In [6] and [5] this same result was obtained (up to the uniqueness), but the relationships between the parameter, b_1 , and b_2 were obtained in a form more convenient for explicit numerical calculation.

(The author is indebted to the referee for calling his attention to reference [9]. In this paper Kühnau proved the bound $|b_2| < .58 \dots$ for the class of elliptic univalent functions. Jenkins [8] has shown that this class and the class \mathcal{E} have the same external functions. Thus the bound for $|b_2|$ obtained in [6] has in principle been known since 1965.)

Following the notation in [5], let B be any complex number in the first quadrant which is not real with $0 < B < 1$. Set

$$P(B) = \int_{-1}^1 [(w + B)/(1 - w^2)]^{1/2} dw, \tag{2.1}$$

$$Q(B) = i \int_{-1}^{-B} [(w + B)/(1 - w^2)]^{1/2} dw, \tag{2.2}$$

where the integrations are along the line segments joining the limits. Suppose $\text{Re}\{Q(B)/P(B)\} > 0$. This will be true for $|B| > 1$ with the proper choice of branches of the roots (see [5]). Determine the real C such that

$$\text{Re}\{Q(B)/P(B)\} = Q(C)/P(C), \quad 1 < C < \infty. \tag{2.3}$$

This C is unique as shown in [6]. Set

$$A = P(C)^2/P(B)^2. \tag{2.4}$$

Then the pair

$$b_1 = |A|, \quad b_2 = 2\bar{A}(C - AB), \tag{2.5}$$

lies on the boundary of the b_1, b_2 coefficient domain for \mathcal{E} . The maximum $|b_2|$ for a given b_1 is obtained when B is purely imaginary, say $B = i\mu, \mu > 0$.

3. The functions $P(B)$ and $Q(B)$ can be evaluated in terms of standard elliptic integrals much as in [6] and these can be expanded by the known asymptotic expansions as given in [2]. However, the complications involved make it no more difficult to do the expansion directly. Thus, setting $w = \sin \theta$ in (2.1) and assuming $|B| > 1$,

$$P(B) = B^{1/2} \int_{-\pi/2}^{\pi/2} \left[1 + \frac{\sin \theta}{B} \right]^{1/2} d\theta = B^{1/2} \sum_{\nu=0}^{\infty} \binom{1/2}{\nu} \int_{-\pi/2}^{\pi/2} B^{-\nu} \sin^{\nu} \theta d\theta$$

or

$$\begin{cases} P(B) = \pi B^{1/2} P_1(B), \\ P_1(B) = \sum_{m=0}^{\infty} p_{2m} B^{-2m}, \\ p_{2m} = (-1)^m \binom{1/2}{2m} \binom{-1/2}{m}. \end{cases} \tag{3.1}$$

The expansion for $Q(B)$ is slightly more difficult. Let $z = -w$. Then

$$\begin{aligned} Q(B) &= B^{1/2} \int_1^B \left[\frac{1 - z/B}{1 - 1/z^2} \right]^{1/2} \frac{dz}{z} \\ &= B^{1/2} \int_1^B \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (-1)^{j+k} \binom{1/2}{j} \binom{-1/2}{k} z^{j-2k-1} B^{-j} dz \\ &= B^{1/2} P_1(B) \log B + B^{1/2} \sum_{k=0}^{\infty} \beta_k B^{-k} \end{aligned}$$

where

$$\begin{cases} \beta_{2k+1} = \binom{1/2}{2k+1} \sum_{j=0}^{\infty} \frac{(-1)^j}{2k-2j+1} \binom{-1/2}{j}, \\ \beta_{2k} = (-1)^k \binom{-1/2}{k} \sum_{\substack{j=0 \\ j \neq 2k}}^{\infty} \frac{(-1)^j}{(j-2k)} \binom{1/2}{j} - \binom{1/2}{2k} \sum_{\substack{j=0 \\ j \neq k}}^{\infty} \frac{(-1)^j}{(2k-2j)} \binom{-1/2}{j}. \end{cases} \tag{3.2}$$

The three sums in (3.2) are probably available somewhere in the literature, but after a short, unsuccessful search, it was found easy enough to evaluate them directly. Thus, to find the first, we set $\beta_{2k+1} = \binom{1/2}{2k+1} \varphi_0(1)$ where

$$\varphi_0(x) = \sum_{j=0}^{\infty} \frac{(-1)^j x^{2j}}{(2k-2j+1)} \binom{-1/2}{j}.$$

Then $(d/dx)(x^{-2k-1} \varphi_0(x)) = -x^{-2k-2}(1-x^2)^{-1/2}$. Integrating by making the trigonometric substitution and using the well-known reduction formula gives

$$\varphi_0(x) = c_0 x^{2k+1} + (1-x^2)^{1/2} [A_{2k+1} + A_{2k-1} x^2 + \dots + A_1 x^{2k}]$$

where the A_j are computable constants and c_0 is the constant of integration. Since $\varphi_0(x)$ is even, c_0 must be zero and hence $\varphi_0(1) = 0$. Thus all $\beta_{2k+1} = 0$.

To evaluate the first sum in the formula for β_{2k} set

$$\varphi_1(x) = \sum_{\substack{j=0 \\ j \neq 2k}}^{\infty} \frac{(-1)^j x^j}{(j-2k)} \binom{1/2}{j}.$$

Then $(d/dx)(x^{-2k} \varphi_1(x)) = x^{-2k-1}(1-x)^{1/2} - \binom{1/2}{2k} x^{-1}$ and hence

$$\begin{aligned} \varphi_1(x) &= c_1 x^{2k} + (1-x)^{1/2} [A_{2k} + A_{2k-1} x + \dots + A_1 x^{2k-1}] \\ &\quad + A_0 x^{2k} \log[1 + (1-x)^{1/2}] \end{aligned}$$

where, for $k > 0$,

$$A_0 = A_1 = -2 \binom{1/2}{2k},$$

$$A_j = 2(-1)^j \binom{1/2}{2k} / \binom{-3/2}{j-1}, \quad 0 < j < 2k,$$

$$A_{2k} = -1/2k,$$

while when $k = 0$, $\varphi_1(x) = c_1 + 2(1 - x)^{1/2} - 2 \log[1 + (1 - x)^{1/2}]$.

Since the coefficient of x^{2k} in $\varphi_1(x)$ must be zero, the constant of integration, c_1 , is determined by $c_1 + 2 - 2 \log 2 = 0$ when $k = 0$ and by

$$c_1 + A_0 \log 2 + \sum_{j=1}^{2k} (-1)^j \binom{1/2}{j} A_j = 0$$

when $k > 0$. The value of the desired sum is $\varphi_1(1) = c_1$.

The remaining sum can be evaluated in a similar way by setting

$$\varphi_2(x) = \sum_{\substack{j=0 \\ j \neq k}}^{\infty} (-1)^j \binom{-1/2}{j} \frac{x^{2j}}{(2k - 2j)}.$$

We find

$$\begin{aligned} \varphi_2(x) &= c_2 + \log[1 + (1 - x^2)^{1/2}] \quad \text{when } k = 0 \text{ and} \\ &= c_2 x^{2k} + (1 - x^2)^{1/2} [B_{2k} + B_{2k-2} x^2 + \dots + B_2 x^{2k-2}] \\ &\quad + B_0 x^{2k} \log[1 + (1 - x^2)^{1/2}] \end{aligned}$$

when $k > 0$, where

$$\begin{aligned} B_{2j} &= (-1)^{j+k+1} \binom{-1/2}{k} / \binom{-3/2}{j-1}, \quad j > 1, \\ &= (-1)^k \binom{-1/2}{k}, \quad j = 0, 1. \end{aligned}$$

The final result of the calculations gives

$$\begin{cases} Q(B) = B^{1/2} P_1(B) \log 8B + B^{1/2} Q_1(B), \\ Q_1(B) = \sum_{m=0}^{\infty} q_{2m} B^{-2m} \end{cases} \tag{3.3}$$

where

$$q_0 = -2, \quad q_{2m} = (1/2m - 2\sigma_{2m-1} - \sigma_m) p_{2m}, \quad m > 0,$$

$$\sigma_k = \frac{1}{1 \cdot 2} + \frac{1}{3 \cdot 4} + \dots + \frac{1}{(2k-1)2k}.$$

From these results, the first few terms of the series for $P_1(B)$ and $Q_1(B)$ are easily calculated.

$$\begin{cases} P_1(B) = 1 - \frac{1}{2^4} B^{-2} - \frac{15}{2^{10}} B^{-4} - \frac{105}{2^{14}} B^{-6} - \frac{15015}{2^{22}} B^{-8} + \dots, \\ Q_1(B) = -2 + \frac{1}{2^4} B^{-2} + \frac{47}{2^{11}} B^{-4} + \frac{1097}{3 \cdot 2^{15}} B^{-6} + \frac{329177}{3 \cdot 2^{24}} B^{-8} + \dots. \end{cases} \tag{3.4}$$

4. From (3.1), (3.3), and (3.4),

$$\begin{aligned} \pi Q(B)/P(B) &= \log 8B + Q_1(B)/P_1(B) \\ &= \log 8B - 2 - \frac{1}{2^4} B^{-2} - \frac{21}{2^{11}} B^{-4} - \frac{79}{3 \cdot 2^{13}} B^{-6} - \frac{23005}{2^{24}} B^{-8} + \dots. \end{aligned}$$

To obtain the maximum $|b_2|$, we put $B = i\mu, \mu > 0$. We must then find C satisfying (2.3). Such a C is determined by

$$\log \frac{C}{\mu} - \frac{1}{2^4} C^{-2} - \frac{21}{2^{11}} C^{-4} + \dots = \frac{1}{2^4} \mu^{-2} - \frac{21}{2^{11}} \mu^{-4} + \dots.$$

If $C = \mu C_1(\mu)$ then one easily verifies

$$C_1(\mu) = 1 + \frac{1}{2^3} \mu^{-2} - \frac{1}{2^7} \mu^{-4} + \frac{11}{3 \cdot 2^{10}} \mu^{-6} - \frac{37}{2^{15}} \mu^{-8} + \dots.$$

Putting this into (2.4) gives

$$\begin{aligned} A &= [P(C)/P(B)]^2 = -iC_1(\mu)P_1(C_1(\mu))^2P_1(i\mu)^{-2} \\ &= -i \left[1 - \frac{1}{2^3} \mu^{-2} + \frac{3}{2^7} \mu^{-4} - \frac{55}{3 \cdot 2^{10}} \mu^{-6} + \frac{259}{2^{15}} \mu^{-8} + \dots \right]. \end{aligned}$$

Let $\beta = 1 - |A|$ and invert the series to obtain

$$\begin{aligned} \frac{1}{\mu^2} &= 2^3 \left(\beta + \frac{3}{2} \beta^2 - \frac{14}{3} \beta^3 - \frac{39}{2} \beta^4 + \dots \right), \\ \mu &= 2^{-3/2} \beta^{-1/2} \left(1 - \frac{3}{4} \beta + \frac{305}{3 \cdot 2^5} \beta^2 + \frac{441}{2^7} \beta^3 + \dots \right) \end{aligned}$$

and finally

$$\begin{aligned} b_2 &= 2\bar{A}(C - AB) = 2i(1 - \beta)\mu(C_1(\mu) - 1 + \beta) \\ &= i2^{1/2}\beta^{1/2} \left(1 - \frac{5}{4} \beta + \frac{85}{3 \cdot 2^5} \beta^2 - \frac{55}{3 \cdot 2^7} \beta^3 + \dots \right) \end{aligned}$$

which is the result (1.3) to be proved.

5. The expansions of (3.4) are valid for any B with $|B| > 1$. Hence we could duplicate the calculations of §4 with arbitrary $B, |B| > 1$. The computational difficulties are mainly in solving for C . However one can evaluate the first few terms of the expansions for b_1 and b_2 without much difficulty.

Set $\arg B = \phi$. Then (2.5) gives

$$b_1 = 1 - \frac{\sin^2 \phi}{8|B|^2} - \frac{\sin^2 \phi(43 - 49 \sin^2 \phi)}{256|B|^4} + \dots$$

$$b_2 = \frac{i \sin \phi}{2|B|} \left[1 + \frac{(17 - 25 \sin^2 \phi - 9i \sin \phi \cos \phi)}{32|B|^2} + \dots \right].$$

Again setting $\beta = 1 - b_1$ and inverting one finds

$$b_2 = i2^{1/2}\beta^{1/2} \left[1 - \frac{(9 + \sin^2 \phi + 18i \sin \phi \cos \phi)}{8 \sin^2 \phi} \beta + \dots \right]. \quad (5.1)$$

Unfortunately, this formula is only useful for large $|B|$. Thus (5.1) defines only a small part of the boundary of the b_2 coefficient region. In particular it is not valid at what are called the "critical points" in [5].

Formula (5.1) is sufficient to show that the "tip" of the b_2 region is approximately of quadratic (parabolic) character near $b_2 = i\sqrt{2}\beta^{1/2}$.

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