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THE HAHN DECOMPOSITION THEOREM

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ABSTRACT. Let $(X, \mathcal{Q}, \mu)$ be a signed measure on the σ-algebra $\mathcal{Q}$ of subsets of X. We give a very short proof of the Hahn decomposition theorem, namely, that X can be partitioned into two subsets P and N such that P is positive: $\mu(E) \geqslant 0$ for every $E \subset P$, and N is negative: $\mu(E) \leqslant 0$ for every $E \subset N$.

A *signed measure* μ on the measurable space $(X, \mathcal{Q})$ is an extended real valued set function defined on the sets of the σ-algebra $\mathcal{Q}$ and satisfying

(i) $\mu(\varnothing) = 0$.

(ii) μ assumes, at most, one of the values $+\infty, -\infty$.

(iii) $\mu(\bigcup E_n) = \sum \mu(E_n)$ for any sequence of disjoint measurable sets E_n.

Condition (iii) yields immediately the following:

(iii′) If $A_n \searrow$ and $\mu(A_1) \neq \pm\infty$, then $\lim \mu(A_n) = \mu(\bigcap A_n)$.

In what follows we shall suppose that

(ii′) $+\infty$ is the infinite value omitted by μ.

We say that a set $P \in \mathcal{Q}$ is *positive* (with respect to the signed measure μ) if $\mu(E) \geqslant 0$ for every measurable $E \subset P$. Similarly, N is *negative* if $\mu(E) \leqslant 0$ for every $E \subset N$.

LEMMA. *Every $A \in \mathcal{Q}$ with $\mu(A) \neq -\infty$ contains a positive set P such that $\mu(P) \geqslant \mu(A)$.*

PROOF. We first show that to every $\varepsilon > 0$ there corresponds $A_\varepsilon \subset A$ such that $\mu(A_\varepsilon) \geqslant \mu(A)$ and $B \subset A_\varepsilon \Rightarrow \mu(B) > -\varepsilon$. For otherwise, inductively, there is a sequence $B_1 \subset A, \ldots, B_k \subset A \setminus (B_1 \cup \cdots \cup B_{k-1}), \ldots$ such that $\mu(B_k) \leqslant -\varepsilon$. Put $B = \bigcup B_k$. Since the B_k are disjoint, then $\mu(B) = -\infty$, $\mu(A \setminus B) = \mu(A) - \mu(B) = +\infty$, against (ii′). Now choosing $\varepsilon_r \to 0$, $A_{\varepsilon_n} \searrow$ and putting $P = \bigcap A_{\varepsilon_n}$ we see that P is positive and by (iii′) $\mu(P) \geqslant \mu(A)$.

THE HAHN DECOMPOSITION THEOREM. *Let μ be a signed measure on the measurable space $(X, \mathcal{Q})$. Then X can be partitioned into a positive set P and a negative set N.*

PROOF. Let $s = \sup\{\mu(A): A \in \mathcal{Q}\}$. There is a sequence P_n such that $\mu(P_n) \to s$ and, by the lemma, we may assume the P_n are positive. Putting $P = \bigcup P_n$ we have $\mu(P) = s$ and P is positive. But $N = X \setminus P$ is negative, for if $E \subset N$ and $\mu(E) > 0$ then $\mu(P \cup E) > s$ which is impossible.

The proof is now complete.

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