PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 80, Number 3, November 1980

THE MEAN-VALUE ITERATION FOR SET-VALUED MAPPINGS

PETER K. F. KUHFITTIG

ABSTRACT. In this note Krasnoselskii's iteration procedure

$$x_{n+1} = \tfrac{1}{2}(I + T)x_n$$

is extended to certain classes of set-valued mappings.

1. Introduction. Let C be a convex subset of a Banach space B and T a self-mapping of C and consider the following iteration process of a type introduced by Mann [7]: for an arbitrary starting point $x_0 \in C$

$$x_{n+1} = (1 - c_n)x_n + c_n Tx_n, \qquad n = 0, 1, 2, \ldots, \tag{$*$}$$

where $c_n \in [a, b]$ for $0 < a < b < 1$. The special case $c_n = \frac{1}{2}$ for all n was first introduced by Krasnoselskii [5], who showed that the sequence converges to a fixed point of T if T is nonexpansive, B uniformly convex, and C compact. This result remains valid if $c_n = \alpha$, $0 < \alpha < 1$ (Schaefer [12]). Moreover, it is sufficient to assume that B is strictly convex (Edelstein [3]). Retaining uniform convexity, Browder and Petryshyn [1] assumed C to be closed and T demicompact. Under the latter conditions, the sequence $(*)$ converges to a fixed point of T if T is merely continuous and quasinonexpansive, that is, nonexpansive about its set of fixed points, assumed nonempty. (See Corollary which follows.) The iteration $(*)$ has been investigated by Senter and Dotson [13].

In this paper we shall consider an analogous iteration for a mapping T: $C \to K(C)$, where $K(C)$ is the family of nonempty compact subsets of C. It is assumed that one fixed point z is known and that T is nonexpansive about this point, that is, for all $x \in C$

$$D(Tx, Tz) \leq \|x - z\|,$$

where D is the Hausdorff metric on $K(C)$. The iteration procedure is designed to generate additional fixed points.

Regarding the existence of fixed points, it was shown by Lim [6] that if C is a convex closed and bounded subset of a uniformly convex Banach space, then a nonexpansive mapping T: $C \to K(C)$ has a fixed point. This result has recently been extended by Downing and Kirk [2].

2. The sequence. Let $z \in Tz$ be the known fixed point. Since Tx is compact and D the Hausdorff metric, we can find for every $x \in C$ a point $p_x \in Tx$ such that

$$\|z - p_x\| \leq D(Tz, Tx).$$

Received by the editors September 5, 1979 and, in revised form, December 3, 1979.
AMS (MOS) subject classifications (1970). Primary 47H10; Secondary 54C60.
Key words and phrases. Fixed points, set-valued mappings, iteration.

Using this information, suppose we construct a sequence $\{x_n\}$ in C as follows: let $x_0 \in C$ and $p_0 \in Tx_0$. Next let

$$x_1 = (1 - c_0)x_0 + c_0 p_0,$$

where $c_0 \in [a, b]$, $0 < a < b < 1$. Then we can find $p_1 \in Tx_1$ such that

$$\|z - p_1\| \leqslant D(Tz, Tx_1)$$

by the prior comments. Now let

$$x_2 = (1 - c_1)x_1 + c_1 p_1.$$

Since Tx_2 is compact, we can find $p_2 \in Tx_2$ such that

$$\|z - p_2\| \leqslant D(Tz, Tx_2).$$

Continuing in this manner

$$x_{n+1} = (1 - c_n)x_n + c_n p_n, \qquad n = 0, 1, 2, \ldots,$$

where $c_n \in [a, b]$ for $0 < a < b < 1$, $p_n \in Tx_n$, and

$$\|z - p_n\| \leqslant D(Tz, Tx_n).$$

Since T is not even assumed to be quasinonexpansive, we do require continuity in the following sense.

DEFINITION 1. A mapping T: $C \to K(C)$ is *continuous* if for any sequence $\{y_n\}$ in C, $y_n \to y$ implies that $Ty_n \to Ty$.

DEFINITION 2 (PETRYSHYN [10]). A mapping U: $C \to B$ of a subset C of a Banach space B into B is said to be *demicompact* if whenever $\{x_n\} \subset C$ is a bounded sequence and $\{x_n - Ux_n\}$ is a convergent sequence, then there exists a subsequence $\{x_{n_i}\}$ which is convergent.

For set-valued mappings we have the following analogous definition.

DEFINITION 3. A mapping T: $C \to K(C)$ will be called *demicompact* if whenever $\{x_n\} \subset C$ is a bounded sequence and $\{\text{dist}(x_n, Tx_n)\}$ is a convergent sequence, then there is a subsequence $\{x_{n_i}\}$ which is convergent.

In the proof of the first theorem we are going to need the following two lemmas.

LEMMA 1 (SCHAEFER [12]). *Let B be a uniformly convex Banach space. Then for $\varepsilon > 0$, $d > 0$, and $\alpha \in (0, 1)$ the inequalities $\|x\| \leqslant d$, $\|y\| \leqslant d$, and $\|x - y\| \geqslant \varepsilon$ imply that*

$$\|(1 - \alpha)x + \alpha y\| \leqslant [1 - 2\delta(\varepsilon/d)\min(1 - \alpha, \alpha)]d;$$

δ is strictly increasing.

LEMMA 2 (NADLER [9]). *Let $\{A_n\}$ be a sequence of sets in $K(C)$ and suppose $\lim_{n\to\infty} D(A_n, A_0) = 0$, where $A_0 \in K(C)$. Then if $x_n \in A_n$, $n = 1, 2, \ldots,$ and if $\lim_{n\to\infty} x_n = x_0$, it follows that $x_0 \in A_0$.*

3. Results.

THEOREM 1. *Let C be a nonempty convex closed subset of a uniformly convex Banach space B. If T: $C \to K(C)$ is a continuous demicompact mapping which is nonexpansive about a known fixed point z, then for the sequence $\{x_n\}$ defined previously*, (a) *there exists a subsequence $\{x_{n_i}\}$ converging to a fixed point of T and*

(b) *every cluster point of* $\{x_n\}$ *is a fixed point of* T. (*In particular, every convergent subsequence of* $\{x_n\}$ *converges to a fixed point.*)

PROOF. The first step is to show that for the sequence $\{x_n\}$ constructed previously

$$\|x_n - p_n\| \to 0 \quad \text{as } n \to \infty.$$

If not, then there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and a number $\varepsilon > 0$ such that

$$\|x_{n_j} - p_{n_j}\| \geqslant \varepsilon. \tag{1}$$

Since $p_n \in Tx_n$,

$$\|z - p_n\| \leqslant D(Tz, Tx_n) \leqslant \|z - x_n\|. \tag{2}$$

Then by (1), (2) and Lemma 1 there exists

$$\delta = \delta(\varepsilon/\|z - x_{n_j}\|) > 0$$

such that

$$\begin{aligned}\|z - x_{n_j+1}\| &= \|z - (1 - c_{n_j})x_{n_j} - c_{n_j}p_{n_j}\|\\ &= \|(1 - c_{n_j})(z - x_{n_j}) + c_{n_j}(z - p_{n_j})\|\\ &\leqslant (1 - \delta\gamma)\|z - x_{n_j}\|,\end{aligned}$$

where $\gamma = 2\min(1 - c_{n_j}, c_{n_j})$. From

$$\|z - x_{n+1}\| = \|(1 - c_n)(z - x_n) + c_n(z - p_n)\| \leqslant \|z - x_n\|, \tag{3}$$

the sequence $\{\|z - x_n\|\}$ is nonincreasing, and since δ is strictly increasing, the sequence

$$\{\delta(\varepsilon/\|z - x_{n_j}\|)\}$$

in nondecreasing. Since we also have

$$\|z - x_{n_j}\| \leqslant \|z - x_{n_{j-1}+1}\| \leqslant (1 - \delta\gamma)\|z - x_{n_{j-1}}\|$$

for

$$\delta = \delta(\varepsilon/\|z - x_{n_{j-1}}\|)$$

and

$$\gamma = 2\min(1 - c_{n_{j-1}}, c_{n_{j-1}}),$$

it follows that

$$\|z - x_{n_j}\| \to 0 \quad \text{as } j \to \infty.$$

By (2) $\|z - p_{n_j}\| \to 0$, whence $\|x_{n_j} - p_{n_j}\| \to 0$ as $j \to \infty$, contradicting statement (1). Hence

$$\|x_n - p_n\| \to 0 \quad \text{as } n \to \infty, \tag{4}$$

which was to be shown.

It now follows from (4) that $\operatorname{dist}(x_n, Tx_n) \to 0$ as $n \to \infty$. Moreover, by (3), $\{x_n\}$ is a bounded sequence. So by demicompactness there exists a subsequence $\{x_{n_i}\}$ of

$\{x_n\}$ such that $x_{n_i} \to z_0 \in C$. Also, from

$$\|p_{n_i} - z_0\| \leq \|p_{n_i} - x_{n_i}\| + \|x_{n_i} - z_0\|,$$

we have that $p_{n_i} \to z_0$. But $Tx_{n_i} \to Tz_0$ by continuity. Consequently, since $p_{n_i} \in Tx_{n_i}$, $z_0 \in Tz_0$ by Lemma 2.

Finally, if w_0 is a cluster point of $\{x_n\}$, there exists a subsequence converging to w_0, which is a fixed point by the above argument. This completes the proof.

Recall from §2 that $p_x \in Tx$ is a point for which $\|z - p_x\| \leq D(Tz, Tx)$. Suppose for every such $p_x \in Tx$ and $p_y \in Ty$, $T: C \to K(C)$ satisfies the condition

$$D(Tx, Ty) \leq \alpha\|x - p_x\| + \beta\|y - p_y\| \qquad \text{(A)}$$

for all $x, y \in C$ and $\alpha, \beta \in [0, \infty)$.

Then if $\alpha = \beta \in [0, \frac{1}{2}]$ and if T is a point-to-point mapping, T is a Kannan mapping, first introduced by Kannan [4]. Such mappings have been studied extensively.

If p_w is chosen (without reference to the fixed point z) so that $\|w - p_w\| = \text{dist}(w, Tw)$ and if $\alpha = \beta = \frac{1}{2}$, then T becomes a set-valued Kannan mapping. Such mappings were studied by Shiau, et al. [14], [15]. Clearly, any Kannan mapping is of Type A.

THEOREM 2. *If T in Theorem 1 is of Type* A, *then $Tx_n \to Tz_0$, where $z_0 \in Tz_0$.*

PROOF. For every $\varepsilon > 0$ there exists $N > 0$ such that

$$D(Tx_n, Tx_m) \leq \alpha\|x_n - p_n\| + \beta\|x_m - p_m\| < \varepsilon$$

for all $m, n > N$, so that $\{Tx_n\}$ is a Cauchy sequence. Since C is complete, $(K(C), D)$ is complete (Michael [8]). Hence $Tx_n \to L \in K(C)$. Since $x_{n_i} \to z_0$, $Tx_{n_i} \to Tz_0$, so that $L = Tz_0$.

The result of Theorem 2 is, in one sense, the best possible. For if T is a nonexpansive set-valued mapping, then the natural analogue of Krasnoselskii's procedure is the following: let $x_0 \in C$, $q_0 \in Tx_0$, and $x_1 = \frac{1}{2}x_0 + \frac{1}{2}q_0$. Now choose $q_1 \in Tx_1$ such that

$$\|q_1 - q_0\| \leq D(Tx_1, Tx_0).$$

In general, $x_{n+1} = \frac{1}{2}x_n + \frac{1}{2}q_n$, where $q_n \in Tx_n$ and

$$\|q_n - q_{n-1}\| \leq D(Tx_n, Tx_{n-1}),$$

whence

$$\|q_n - q_{n-1}\| \leq \|x_n - x_{n-1}\|.$$

This construction fails, however, as can be seen from the mapping $T: R \to K(R)$ defined by $Tx = [x - 1, x + 1]$. If for $Tx_n = [x_n - 1, x_n + 1]$, we choose $q_n = x_n + 1$, then the resulting sequence has no convergent subsequence.

If T is a point-to-point mapping, then $q_n = Tx_n$, and $\{x_n\}$ reduces to Krasnoselskii's iteration. Now suppose T is continuous, demicompact, and quasinonexpansive with a nonempty set F of fixed points. Returning to the sequence (*), if $z \in F$, then

$$\|z - Tx_n\| \leq \|z - x_n\|$$

and, by the proof of Theorem 1, there exists a subsequence $\{x_{n_i}\}$ for which $x_{n_i} \to z_0 = Tz_0$. Since $\{\|z_0 - x_n\|\}$ is clearly nonincreasing, $x_n \to z_0$. This proves the following

COROLLARY. *Let C be a nonempty convex closed subset of a uniformly convex Banach space. If T: $C \to C$ is a continuous demicompact quasinonexpansive mapping with a nonempty set of fixed points, then the sequence* (*) *converges to a fixed point of* T.

This result is similar to Theorem 1.1′ in **[11]** and Theorem 2 in **[13]**.

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DEPARTMENT OF MATHEMATICS, MILWAUKEE SCHOOL OF ENGINEERING, MILWAUKEE, WISCONSIN 53201