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AN IMPROVED ESTIMATE FOR THE BLOCH NORM OF FUNCTIONS IN DOOB'S CLASS

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ABSTRACT. For any fixed $0 < \rho < 2\pi$, $\mathfrak{D}(\rho)$ is the family of all holomorphic functions in Δ which satisfy (i) $f(0) = 0$, and (ii) $\underline{\lim}_{z\to\tau} |f(z)| \geqslant 1$, for all τ lying on some arc $\Gamma_f \subseteq \partial\Delta$ with arclength $|\Gamma_f| \geqslant \rho$. We showed that for each $f \in \mathfrak{D}(\rho)$ there exists a point $z_f \in \Delta$ at which

$$|f'(z_f)|\left(1 - |z_f|^2\right) \geqslant \frac{2}{e}\,\frac{\sin(\pi - (\rho/2))}{(\pi - (\rho/2))}.$$

In this paper we improve this estimate by replacing the quantity $\pi - (\rho/2)$ with a value $\theta(\rho)$ which lies between 0 and $\pi - (\rho/2)$ and so improves the estimate. The value $\theta(\rho)$ is defined as the (unique) solution in this interval of the equation $F_\rho(\theta) = \log(\cot(\rho/4)\cot(\theta/2)) - \theta/\sin\theta = 0$.

1. In a series of papers ([4]–[8]) J. L. Doob introduced a family $\mathfrak{D}(\rho)$ of holomorphic functions in the unit disc Δ, whose boundary we denote by $\partial\Delta$. For any fixed $0 < \rho < 2\pi$, $\mathfrak{D}(\rho)$ is the family of all holomorphic functions in Δ which satisfy (i) $f(0) = 0$, and (ii) $\underline{\lim}_{z\to\tau} |f(z)| \geqslant 1$, for all τ lying on some arc $\Gamma_f \subseteq \partial\Delta$ with arclength $|\Gamma_f| \geqslant \rho$. Doob posed the question as to whether the set of Bloch norms $\{\|f\|_{\mathfrak{B}} = \sup_{z\in\Delta} |f'(z)|(1 - |z|^2)\}_{f\in\mathfrak{D}(\rho)}$ has a positive lower bound. We showed in [10] that for each $f \in \mathfrak{D}(\rho)$ there exists a point $z_f \in \Delta$ at which

$$|f'(z_f)|\left(1 - |z_f|^2\right) \geqslant \frac{2}{e}\,\frac{\sin(\pi - (\rho/2))}{(\pi - (\rho/2))}. \tag{1.0}$$

In this paper we improve this estimate by replacing the quantity $\pi - (\rho/2)$ with a value $\theta(\rho)$ which lies between 0 and $\pi - (\rho/2)$. The value $\theta(\rho)$ is defined as the (unique) solution in this interval of the equation

$$F_\rho(\theta) = \log(\cot(\rho/4)\cot(\theta/2)) - \frac{\theta}{\sin\theta} = 0.$$

Functions in $\mathfrak{D}(\rho)$ produce upper estimates to the various Bloch constants [2]. For f holomorphic in Δ set $b(f) = \sup\{r|$ there exists a domain $\Delta_1 \subseteq \Delta$ such that f is univalent on Δ and $f(\Delta_1)$ contains a disc of radius $r\}$. If $\mathfrak{B}$ denotes the family of holomorphic functions in Δ normalized by $|f'(0)| \geqslant 1$, then the Bloch constant B is defined as

$$B = \inf b(f), \qquad f \in \mathfrak{B}.$$

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If $\mathfrak{B}_s$ denotes the subfamily of $\mathfrak{B}$ of all univalent functions then $B_s = \inf b(f)$, $f \in \mathfrak{B}_s$. It is known that $\sqrt{3}/4 < B < .472$; $.544 < B_s < .658$. These lower estimates are due to Heins [9] and Landau [11], respectively, while the upper estimates are due to Ahlfors and Grunsky [1] and R. Robinson [13], respectively. With no loss of generality the normalization $|f'(0)| \geqslant 1$ can be relaxed. If L is a Möbius transformation of Δ onto Δ taking 0 into z then we have both $b(f \circ L) = b(f)$ and $(f \circ L)'(0) = (1 - |z|^2)|f'(z)|$. So we enlarge $\mathfrak{B}$ (and $\mathfrak{B}_s$) by replacing $|f'(0)| \geqslant 1$ by $|f'(z_f)|(1 - |z_f|^2) \geqslant 1$, for some $z_f \in \Delta$. The constants B and B_s remain unchanged. From our previous results as stated in (1.0) we see that $f \in \mathfrak{D}(\rho)$ implies

$$\left(\frac{e}{2}\frac{(\pi - (\rho/2))}{\sin(\pi - (\rho/2))}\right)f \in \mathfrak{B},$$

and so upper estimates for B (and B_s) can be obtained from functions in $\mathfrak{D}(\rho)$. Any improvement in the constant $(e/2)(\pi - (\rho/2))/\sin(\pi - (\rho/2))$ should be of some interest.

2. Main result. If $A \subseteq \Delta$, let ∂A denote the topological boundary of A. If $\partial A \cap \partial\Delta$ contains an arc Γ then, as usual, let $\omega(z, \Gamma, A)$ denote the harmonic measure at z of Γ relative to A. If $A = \Delta$ then we define the lens-shaped domain in Δ by

$$S(\alpha, \Gamma) = \{z \in \Delta | \omega(z, \Gamma, \Delta) > (\pi - \alpha)/\pi\}, \qquad 0 < \alpha < \pi.$$

It is easy to show that $\partial S(\alpha, \Gamma) \cap \Delta$ makes the angle α with $\partial\Delta$. If $\alpha = \pi - (\rho/2)$ then $\partial S(\alpha, \Gamma)$ contains the origin. The proof of the main result is based upon a sharpened form of the Lehto-Virtanen differential two constant theorem [12] due to S. Dragosh and D. C. Rung [3]. For completeness we state a less general version suitable for our needs. It is similar to the version used in [10].

THEOREM D-R. *Let f be meromorphic in Δ. Fix a domain of the form $S(\alpha, \Gamma)$ and suppose*

(i) $\sup_{z \in S(\alpha,\Gamma)} |f(z)| = M < \infty$;

(ii) *there exists a point $q \in \partial S(\alpha, \Gamma) \cap \Delta$ at which $|f(q)| = M$;*

(iii) *for each $\tau \in \Gamma$, $\overline{\lim}_{z \to \tau} |f(z)| \leqslant m < M$.*

Then

$$|f'(q)|(1 - |q|^2) \geqslant \left(\frac{2 \sin \alpha}{\alpha}\right) M \log \frac{M}{m}. \tag{2.0}$$

In preparation for the main result we need a few more details. For any fixed $0 < \rho < 2\pi$, denote the (unique) root of $\log[\cot(\rho/4)\cot(\theta/2)] - \theta/\sin\theta = 0$ lying in $(0, \pi - (\rho/2))$ by $\theta(\rho)$. On this interval the first term of the equation decreases from $+\infty$ to 0 while the left side increases from 1 to $(\pi - \rho/2)/\sin(\pi - (\rho/2))$. Given an arc $\Gamma \subseteq \partial\Delta$, with midpoint τ, let r_τ denote the radius to τ and set $\hat{S}(\alpha, \Gamma) = S(\alpha, \Gamma) \cup r_\tau$, $0 < \alpha < \pi$.

Theorem 1. *Suppose $f \in \mathfrak{D}(\rho)$ for some fixed ρ, $0 < \rho < 2\pi$, and some arc $\Gamma_f \subseteq \partial\Delta$. Then there exists at least one point $z_f \in \hat{S}(\theta(\rho), \Gamma_f)$ at which*

$$|f'(z_f)|\left(1 - |z_f|^2\right) > \frac{2}{e}\,\frac{\sin\theta(\rho)}{\theta(\rho)}. \tag{2.1}$$

Proof. We may suppose Γ_f is symmetric about $z = 1$ so that r_τ is the segment $0 < x < 1$. To the contrary suppose for all $z \in \hat{S}(\theta(\rho), \Gamma_f)$

$$|f'(z)|\left(1 - |z|^2\right) < \frac{2}{e}\,\frac{\sin\theta(\rho)}{\theta(\rho)}. \tag{2.2}$$

Let x^* be the intersection of $\partial S(\theta(\rho), \Gamma_f)$ and $[0, 1)$. We are going to show first that (2.2) and Theorem D-R imply $|f(x^*)| > 1/e$; and secondly a straightforward estimate from (2.2) implies $|f(x^*)| < 1/e$. To show the first inequality we put $g = 1/f$ and let Γ_θ denote the subarc of Γ_f with endpoints $e^{-i\theta}$ and $e^{i\theta}$, $0 < \theta < \rho/2$. Put $M(\theta) = \sup|g(z)|$, $z \in S(\theta(\rho), \Gamma_\theta)$. Choose any value θ for which $M(\theta)$ is finite. Apply Theorem D-R to g on the domain $S(\theta(\rho), \Gamma_\theta)$. In this situation $m = 1$ and $M = M(\theta)$ and so we conclude that for some $q_\theta \in \partial S(\theta(\rho), \Gamma_\theta) \cap \Delta$

$$|g'(q_\theta)|\left(1 - |q_\theta|^2\right) > \frac{2\sin(\theta(\rho))}{\theta(\rho)} M(\theta)\log M(\theta). \tag{2.3}$$

If we remember $g = 1/f$ then (2.3) together with assumption (2.2) gives

$$\frac{2\sin\theta(\rho)}{\theta(\rho)}\,\frac{\log M(\theta)}{M(\theta)} < |f'(q_\theta)|\left(1 - |q_\theta|^2\right) < \frac{2}{e}\,\frac{\sin\theta(\rho)}{\theta(\rho)},$$

or

$$\frac{\log M(\theta)}{M(\theta)} < \frac{1}{e}. \tag{2.4}$$

In the interval $[1, \infty)$ the function $\log x/x$ has a single maximum value of $1/e$ at $x = e$. The finite (and infinite) values of $M(\theta)$ form a connected set, which because of (2.4) lies either in $[1, e)$ or $(e, \infty]$. But $f \in \mathfrak{D}(\rho)$ implies that $M(\theta)$ is close to 1 for small values of θ and so we conclude that $M(\theta) < e$ for *all* values of $0 < \theta < \rho/2$. In particular $1/|g(x^*)| = |f(x^*)| > 1/e$. In the other direction we estimate $f(x^*)$ by integrating $f'(x)$ along the interval $[0, x^*]$. Under assumption (2.2) and remembering that $f(0) = 0$ we have

$$|f(x^*)| < \int_0^{x^*}|f'(x)|\,dx < \frac{2}{e}\,\frac{\sin\theta(\rho)}{\theta(\rho)}\int_0^{x^*}\frac{dx}{1 - x^2}$$
$$= \frac{1}{e}\,\frac{\sin\theta(\rho)}{\theta(\rho)}\log\left(\frac{1 + x^*}{1 - x^*}\right). \tag{2.5}$$

To solve for x^* in terms of $\theta(\rho)$ and ρ recall that $\partial S(\theta(\rho), \Gamma_f) \cap \Delta$ is part of a circle making an angle $\theta(\rho)$ with $\partial\Delta$ at $e^{-i\rho/2}$ and $e^{i\rho/2}$. Using the law of sines we calculate x^* as the difference between the distance of the center of the circle determined by $\partial S(\theta(\rho), \Gamma_f)$ from the origin, and the radius of this circle. We obtain that

$$x^* = \frac{\sin(\theta(\rho)) - \sin(\rho/2)}{\sin(\theta(\rho) - (\rho/2))}.$$

A routine use of trigonometric identities shows that

$$\frac{1 + x^*}{1 - x^*} = \cot\left(\frac{\theta(\rho)}{2}\right)\cot(\rho/4). \tag{2.6}$$

Because $\theta(\rho)$ was chosen so that

$$\frac{\sin \theta(\rho)}{\theta(\rho)} \log\left[\cot\left(\frac{\theta(\rho)}{2}\right)\cot(\rho/4)\right] = 1$$

(2.5) and (2.6) show that $f(x^*) < 1/e$. Thus a contradiction is reached and (2.1) is established. Because $0 < \theta(\rho) < \pi - (\rho/2)$ the monotonicity of $\sin x/x$ shows that (2.1) is a better estimate than (1.0). In fact it is a *much* stronger result especially for small values of ρ. For example when $\rho = \pi$ the constant in (1.0) is $4/e\pi \sim .468$, while in (2.1) it is $\sim .684$; if $\rho = \pi/10$ then (1.0) gives a constant of $\sim .08$ while (2.1) gives $\sim .345$. As $\rho \to 2\pi$, $\theta(\rho) \to 0$ and $(2/e)\sin \theta(\rho)/\theta(\rho) \to 2/e$. In this asymptotic case the constant $2/e$ is best possible as was pointed out in **[10]**. Whether $(2/e)\sin \theta(\rho)/\theta(\rho)$ is best possible in general we do not know. In Table 1 we give various values of ρ, $\theta(\rho)$ and $\kappa(\rho) = (2/e)\sin \theta(\rho)/\theta(\rho)$. It is easy to generate extensive values of $\kappa(\rho)$ with any computer. Numbers in the table have been rounded to three places.

TABLE 1

ρ	$\theta(\rho)$	$\kappa(\rho)$
π	.657	.684
$4\pi/5$	.839	.653
$3\pi/5$	1.06	.606
$2\pi/5$	1.33	.538
$\pi/5$	1.70	.429
$\pi/50$	2.35	.224

The convergence of $\kappa(\rho)$ to 0 with ρ is rather slow. We have not been able to use functions belonging to the family $\mathfrak{D}(\rho)$ to obtain any improvement in the upper estimates for B and B_S.

In Theorem 1 the normalization $f(0) = 0$ can be replaced by $f(a) = 0$ to produce a slightly more general theorem.

THEOREM 2. *Let f be holomorphic in Δ and suppose $f(a) = 0$, $a \in \Delta$. Suppose further for some arc $\Gamma_f \subseteq \partial\Delta$ we have, for all $\tau \in \Gamma_f$, $\underline{\lim}_{z\to\tau} |f(z)| > 1$. Then there exists a point $z_f \in S(\pi(1 - \omega(a, \Gamma_f, \Delta)), \Gamma_f)$ at which*

$$|f'(z_f)|\left(1 - |z_f|^2\right) > \kappa\left(2\pi\omega(a, \Gamma_f, \Delta)\right).$$

PROOF. It is easy to see that $g(\zeta) = f((\zeta + a)/(1 + \bar{a}\zeta))$ is in $\mathfrak{D}(2\pi\omega(a, \Gamma_f, \Delta))$ and of course $|g'(\zeta)|(1 - |\zeta|^2) = |f'(z)|(1 - |z|^2)$, $z = (\zeta + a)/(1 + \bar{a}\zeta)$. An application of Theorem 1 to $g(\zeta)$ proves Theorem 2. (Actually z_f lies in a smaller domain but for simplicity we use the more familiar albeit larger domain $S(\pi(1 - \omega(a, \Gamma_f, \Delta)), \Gamma_f)$.)

We close with several questions on the classes $\mathfrak{D}(\rho)$. Can one say anything about the boundary behavior of f away from the arc Γ_f on which $|f| > 1$? By applying Fatou's Theorem to $1/f$ in a neighborhood of T_f we see that f has angular limits almost everywhere on Γ_f. Can the condition $\underline{\lim}_{z \to \Gamma_f} |f(z)| > 1$ be relaxed to allow this lower limit only for certain approaches to Γ_f? And lastly, is it possible to allow Γ to be the union of finitely many arcs–perhaps symmetrically arranged on $\partial\Delta$–and to produce a lower estimate for $\|f\|_{\mathfrak{B}}$ which is better than the one obtained by considering the largest subarc of Γ?

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