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ON THE NOTE OF C. L. BELNA

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ABSTRACT. The space of functions of ordered harmonic bounded variation ($OHBV$) has been shown by Belna to contain the space of functions of harmonic bounded variation (HBV) properly. $OHBV$ is a Banach space and HBV is a first category subset. The ordered harmonic variation has continuity properties quite different from those of the harmonic variation. The relationship of these classes to the everywhere convergence of Fourier series is discussed.

In the previous note C. Belna has shown that the inclusion $OHBV \supset HBV$ is proper [2]. Here we discuss the implications of that fact for various problems on everywhere convergence of Fourier series and we will use Belna's example to show an important respect in which the ordered harmonic variation behaves differently from the harmonic variation. We will also show that with a suitable norm $OHBV$ is a Banach space and HBV is a first category subset.

With C. Goffman [3], [4], and [5] we characterized the functions whose Fourier series converge everywhere for every change of variable. A right system of intervals at a point x is a collection of intervals $\{I_{k,n}\}$, $k = 1, 2, \ldots, n = 1, \ldots, n_k$, such that for each k and n, $I_{k,n+1}$ is to the right of $I_{k,n}$, and for each $\delta > 0$ there is an N such that $I_{k,n} \subset (x, x + \delta)$ for $k > N$. A left system is analogously defined. Let $f(b) - f(a) = f([a, b])$. A function is said to be *regulated* if it has right and left limits at each point. A regulated function f has a Fourier series which converges everywhere for every change of variable if and only if $\sum_{n=1}^{n_k} f(I_{k,n})/n \to 0$ as $k \to \infty$ for every right and left system at each point. The class of functions satisfying this property was called GW by A. Baernstein [1] in the paper in which we characterized the *continuous* functions (UGW) for which the Fourier series converges *uniformly* for every change of variable. The condition for UGW is obtained from that for GW by adding the proviso that $\sum_{n=1}^{n_k} f(I_{k,n})/(n_k - n + 1) \to 0$.

The form of the conditions for GW and UGW clearly suggests that one look at ordered collections of intervals of the type that appear in the definition of $OHBV$, but our results [6] and [8] on the convergence of Fourier series of functions of generalized bounded variation seem to require the consideration of *all* collections of intervals instead. The basic result here is that Fourier series of a function in HBV converges everywhere and converges uniformly on any closed interval of points of continuity. Thus $GW \supset HBV$ and $UGW \supset HBV_C$; X_C will denote the continuous functions in class X.

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1. If a function $f \notin OHBV$ on an interval $[a, b]$, it is easy to see that it is not in $OHBV$ on one of the intervals $[a, (a+b)/2]$ and $[(a+b)/2, b]$. Continuing this process we can produce a point p in $[a, b]$ such that $f \notin OHBV$ on any neighborhood of p. It is elementary that on one side of p we must have $f \notin OHBV$ on every interval ending at p. Thus $f \notin UGW$ and we have shown that $UGW \subset OHBV$.

If we examine the example of Belna we see that the first part of condition (b) implies that

$$\sum_{j=1}^{T_k} f(I^*_{\sigma_k+j})/j > 1.$$

Choose $I_n = [a_n, b_n]$ so that $b_{\sigma_k} \to 0$ as $k \to \infty$. We see then that $f \notin GW$. We summarize these facts in the following statement.

PROPOSITION 1. (a) $OHBV_C \supsetneqq UGW \supset HBV_C$.
(b) $OHBV - GW \neq \varnothing$.

An interesting problem here is still unresolved.

Question. Is $GW = HBV$ and $UGW = HBV_C$?

2. If $f \in HBV$, then the *harmonic variation* of f on an interval $[a, b]$, $V_H(f; [a, b])$, is the supremum of sums $\sum |f(I_n)|/n$ where the I_n are nonoverlapping intervals in $[a, b]$. A corresponding point function $V_H(x) = V_H(f; [a, x])$ is defined. It has been shown that $V_H(x)$ is right (left) continuous at x_0 if and only if f is right (left) continuous there [7], [8]. We may define an ordered harmonic variation $V_{OH}(f; [a, b])$ in an analogous manner by adding the requirement of order to $\{I_n\}$.

It is not difficult to see that the continuity of $V_{OH}(x) = V_{OH}(f; [a, x])$ does *not* follow from the continuity of f.

Suppose in the example of Belna that we require $a_{\sigma_k} \to 1/2$. Then $f \equiv 0$ on $[0, 1/2]$ and is continuous at $1/2$. However $V_{OH}(x) = 0$ for $x < 1/2$, but, for $x > 1/2$, $V_{OH}(x) > 1$ by the first part of condition (b).

3. Let us consider $OHBV$ on $[a, b]$ and let $V(f) = V_{OH}(f; [a, b])$. We have the following result.

PROPOSITION 2. *$OHBV$ is a Banach space with norm $\|f\| = |f(a)| + V(f)$.*

It is clear that $OHBV$ is a linear space and that $\|\cdot\|$ is a norm. Convergence in this norm implies uniform convergence. If g is the pointwise limit of a norm convergent sequence $\{f_n\}$, then given $\varepsilon > 0$, there is an N such that $n, m > N$ imply $\sum |f_n(I_k) - f_m(I_k)|/k < \varepsilon$ for every ordered finite sequence of nonoverlapping intervals I_k. Then

$$\sum |g(I_k)|/k < \sum |f_m(I_k)|/k + \varepsilon < V(f_m) + \varepsilon,$$

implying that $g \in OHBV$. Further,

$$V(g - f_n) < \varepsilon + \sum |g(I_k) - f_n(I_k)|/k$$

for some $\{I_k\}$. For $n > N$ and $m_0 > N$ so large that $\Sigma\, |g(I_k) - f_{m_0}(I_k)|/k < \varepsilon$ we have

$$V(g - f_n) \leqslant \varepsilon + \sum |f_n(I_k) - f_{m_0}(I_k)|/k + \sum |f_{m_0}(I_k) - g(I_k)|/k < 3\varepsilon,$$

and so $\| g - f_n \| \to 0$.

We will now show

PROPOSITION 3. *HBV is a first category subset of OHBV.*

Consider the construction of Belna. For any function g define

$$F_m(g) = \sum_{n=1}^{\sigma_{m+1}} g(I_n^*)/n, \qquad m = 1, 2, \ldots .$$

For any m, F_m is a continuous linear functional on $OHBV$, since it is clear that $|F_m(g)| \leqslant m\|g\|$. However

$$F_m(f) = \sum_{k=1}^{m} \sum_{j=1}^{T_k} A_{\sigma_k + j}/(\sigma_k + j) > m$$

by condition (b) part one of Belna's construction. Thus, by the Banach-Steinhaus theorem, the set of g for which $\{F_m(g)\}$ is bounded is of first category. For $g \in HBV$, $\{F_m(g)\}$ is bounded by the harmonic variation of g, implying that HBV is of first category.

REFERENCES

1. A. Baernstein and D. Waterman, *Functions whose Fourier series converge uniformly for every change of variables*, Indiana Univ. Math. J. **22** (1972), 569–576.

2. C. L. Belna, *On ordered harmonic bounded variation*, Proc. Amer. Math. Soc. **80** (1980), 441–444.

3. C. Goffman, *Everywhere convergence of Fourier series*, Indiana Univ. Math. J. **20** (1970), 107–113.

4. C. Goffman and D. Waterman, *Functions whose Fourier series converge for every change of variable*, Proc. Amer. Math. Soc. **19** (1968), 80–86.

5. ______, *A characterization of the class of functions whose Fourier series converge for every change of variable*, J. London Math. Soc. (2) **10** (1975), 69–74.

6. D. Waterman, *On convergence of Fourier series of generalized bounded variation*, Studia Math. **44** (1972), 107–117.

7. ______, *On Λ-bounded variation*, Studia Math. **57** (1976), 33–45.

8. ______, *Fourier series of functions of Λ-bounded variation*, Proc. Amer. Math. Soc. **74** (1979), 119–123.

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