PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 80, Number 3, November 1980

L^p-ESTIMATES FOR MATRIX COEFFICIENTS OF IRREDUCIBLE REPRESENTATIONS OF COMPACT GROUPS

SAVERIO GIULINI AND GIANCARLO TRAVAGLINI[1]

ABSTRACT. The following result is proved.

THEOREM. *Let G be a compact connected semisimple Lie group. For any $p > 0$ there exist two positive numbers A_p and B_p such that (up to equivalence) for any continuous irreducible unitary representation π of G there exists a matrix coefficient a_π of π such that*

$$A_p < d_\pi \int_G |a_\pi|^p < B_p$$

where d_π is the degree of π.

As an application we show the nonexistence of infinite local Λ_q-sets.

1. In this paper we estimate the L^p-norms, for $p > 0$, of certain matrix coefficients of irreducible unitary representations of compact connected semisimple Lie groups. Our estimates show that there exist matrix coefficients with L^p-norms of the order of $d^{-1/p}$ where d is the degree of the irreducible representation. This shows that the ratios between different L^p-norms on the spaces of matrix coefficients of an irreducible representation are unbounded as the degree of the representation increases. In particular we obtain a result on the nonexistence of lacunary sets of irreducible representations which includes earlier results of J. F. Price [4] and D. G. Rider [5] obtained for the special cases SU_2 and SU_n.

We express our appreciation to Professor S. Helgason for some helpful conversations. Moreover we would like to thank the referee for a simplification of our original argument.

2. Let G denote a compact connected semisimple Lie group. The dual object Σ of G is taken to be a maximal set of pairwise inequivalent continuous irreducible unitary representations of G. U_λ denotes a representative of the class $\lambda \in \Sigma$; χ_λ and d_λ are the character and the dimension of λ respectively. Let T denote a maximal torus of G. $\mathfrak{g}_{\mathbf{C}}$ and $\mathfrak{h} = \mathfrak{t}_{\mathbf{C}}$ are the complexifications of the Lie algebras of G and T respectively. If Δ is the set of the roots of $(\mathfrak{g}_{\mathbf{C}}, \mathfrak{h})$ we choose in Δ a system P of positive roots and the associated system of simple roots. The dual object of G may be identified with the semilattice of the dominant weights of G. We order the weights letting $\lambda_2 \preccurlyeq \lambda_1$ if $\lambda_1 - \lambda_2$ is a sum of simple roots with nonnegative integral coefficients (see [6, p. 314]). We take $\lambda_2 \prec \lambda_1$ if $\lambda_2 \preccurlyeq \lambda_1$ and $\lambda_1 \neq \lambda_2$.

Received by the editors June 21, 1979.

AMS (MOS) subject classifications (1970). Primary 43A40, 22E46; Secondary 43A46.

Key words and phrases. Semisimple Lie groups, representations, weights.

[1]This work was partially supported by the C.N.R.

For a weight λ we fix a basis for the vector space on which U_λ acts and let $u_{ij}^{(\lambda)}(g)$ denote the (i,j) matrix entry of $U_\lambda(g)$. Let T_λ be the space of all finite linear combinations of the entry functions $u_{ij}^{(\lambda)}(g)$.

Moreover, for any λ, ξ_λ: $T \to \mathbf{C}$ is the function defined by $\xi_\lambda(\exp X) = e^{\lambda(X)}$ where $X \in \mathfrak{t}$.

3. We prove the following result.

THEOREM. *Let G denote a compact connected semisimple Lie group with dual object Σ. For any $p > 0$ there exist two positive numbers A_p and B_p such that for any $\lambda \in \Sigma$ there exist $U_\lambda \in \lambda$ and a diagonal matrix coefficient a_λ of U_λ such that*

$$A_p < d_\lambda \int_G |a_\lambda|^p < B_p.$$

PROOF. Since G is connected and semisimple, $\exp(\mathfrak{g})$ generates the whole of G and $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$. Therefore, if π is any finite dimensional unitary representation of G in a complex Hilbert space H_{d_π} with inner product $(\cdot, \cdot)$ and $d\pi$ is its derived homomorphism, we have $\operatorname{tr}(d\pi(X)) = 0$ whenever $X \in \mathfrak{g}$, hence $\det(\pi(g)) = 1$ whenever $g \in G$. Moreover, if $\lambda_1, \ldots, \lambda_{d_\pi}$ are the weight of the representation π, there exists a basis of weight vectors $e_1, \ldots, e_{d_\pi}$ in H_{d_π} such that

$$\pi(t)e_i = \xi_{\lambda_i}(t), \qquad t \in T, i = 1, \ldots, d_\pi.$$

We have chosen the representative U_λ: $G \to SU_{d_\lambda}$ such that, if $\lambda = \lambda_1$ is the maximal weight of π, for any $t \in T$, $U_\lambda(t)$ is a diagonal matrix and $u_{11}^{(\lambda)}(t) = \xi_\lambda(t)$, $t \in T$.

In particular, if $\omega_1, \ldots, \omega_r$ are the fundamental weights of G, U_{ω_i} has a diagonal entry function ζ_i such that $\zeta_i|_T = \xi_{\omega_i}$.

Let $\lambda = \sum_{i=1}^r s_i \omega_i$; then U_λ admits $E_\lambda = \prod_{i=1}^r \zeta_i^{s_i}$ as a matrix coefficient for the highest weight vector.

For, if π_σ and π_μ are irreducible representations of G with maximal weights σ and μ respectively, the representation with maximal weight $\sigma + \mu$ appears exactly once in the decomposition of $\pi_\sigma \otimes \pi_\mu$ into a direct sum of irreducible representations (see [6, p. 327]).

Let $e_1^{(\sigma)}$ and $e_1^{(\mu)}$ denote maximal weight vectors for π_σ and π_μ respectively. $\pi_\sigma \otimes \pi_\mu$ induces the representation $\pi_{\sigma+\mu}$ (up to equivalence) on the smallest invariant subspace of $H_\sigma \otimes H_\mu$ containing $e_1^{(\sigma)} \otimes e_1^{(\mu)}$. Hence the matrix coefficient from the highest weight vector of $\pi_{\sigma+\mu}$ is

$$\left(e_1^{(\sigma)} \otimes e_1^{(\mu)}, \pi_\sigma \otimes \pi_\mu(g)(e_1^{(\sigma)} \otimes e_1^{(\mu)})\right) = \left(e_1^{(\sigma)}, \pi_\sigma(g)e_1^{(\sigma)}\right)\left(e_1^{(\mu)}, \pi_\mu(g)e_1^{(\mu)}\right).$$

Hence E_λ is a diagonal entry function for U_λ.

Structure theorems allow us to suppose G simply connected.

If $p > 0$ there are integers $m_i \geqslant 0$ $(i = 1, \ldots, r)$ such that

$$2m_i \leqslant ps_i \leqslant 2(m_i + 1). \tag{1}$$

We write $\gamma = \sum_{i=1}^r m_i \omega_i$ and $\beta = \sum_{i=1}^r \omega_i$. Since $\|\zeta_i\|_\infty \leqslant 1$ we have from (1) that

$$|E_{\gamma+\beta}(g)|^2 \leqslant |E_\lambda(g)|^p \leqslant |E_\gamma(g)|^2.$$

Hence

$$d_{\gamma+\beta}^{-1} \leqslant \int_G |E_\lambda(g)|^p \, dg \leqslant d_\gamma^{-1}. \tag{2}$$

From (1) and Weyl's dimension formula we have

$$d_{\gamma+\beta} \leqslant \prod_{\alpha\in P} \frac{\sum_{i=1}^r((p/2)s_i + 2)\langle\omega_i, \alpha\rangle}{\langle\beta, \alpha\rangle} \leqslant (\mathrm{Max}(p, 2))^{\mathrm{card}\, P} \cdot d_\lambda. \tag{3}$$

In the same way

$$d_\lambda \leqslant (2\,\mathrm{Max}(1, 2/p))^{\mathrm{card}\, P} \cdot d_\gamma. \tag{4}$$

The theorem follows from (2), (3) and (4).

4. The above result may also be seen as a theorem of the theory of lacunary sets for compact Lie groups.

DEFINITION. *Let G denote a compact Lie group with dual object Σ. We say that $E \subset \Sigma$ is a local Λ_q-set ($q > 0$) provided that we have for any r, s ($0 < r < s \leqslant q$),*

$$\operatorname*{Sup}_{\substack{\lambda\in E\\ f\in T_\lambda}} \frac{\left\{\int_G |f|^s\right\}^{1/s}}{\left\{\int_G |f|^r\right\}^{1/r}} < \infty.$$

We have

COROLLARY. *A compact connected semisimple Lie group does not admit infinite local Λ_q-sets for any $q > 0$.*

Related results on lacunarity for compact nonabelian groups may be found in [1], [2] and [3].

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ISTITUTO MATEMATICO DELL'UNIVERSITÀ, VIA SALDINI 50, 20133 MILANO, ITALY