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ON A QUESTION CONCERNING COUNTABLY GENERATED z-IDEALS OF $C(X)$

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ABSTRACT. In $[\mathbf{D}_1]$ the following question was asked: is every countably generated z-ideal of $C(X)$ of the form $O^A = \bigcap_{p \in A} O^p$, for some zero-set A of βX? It is proved here that the answer is affirmative when X is normal and first countable; and an example is given, disproving the general conjecture. For terminology and notation see **[GJ]**, $[\mathbf{D}_1]$.

1. Countably generated z-ideals.

1.1. First of all, we observe that the question asked in $[\mathbf{D}_1]$ may be reformulated in a purely algebraic way: that is, it involves $C(X)$ only, and not the underlying space X, which need not even be assumed completely regular.

Their usual definition notwithstanding, z-ideals are purely algebraic objects: see **[GJ**, 4A.5]. And ideals of the form $O^A = \bigcap_{p \in A} O^p$, with A being a *closed subset* of βX, may also be defined algebraically, with no reference to X or βX: it is proved in **[Br]**, in **[Bk]** or in $[\mathbf{D}_2]$ that such ideals are exactly the *pure ideals* of $C(X)$ (we recall that an ideal I of a commutative ring R (with 1) is said to be *pure* if for every $f \in I$ there exists $g \in I$ such that $f = fg$). In $[\mathbf{D}_1$, Lemma 2.1], it is proved that O^A (with A closed in βX) is countably generated if and only if A is a zero-set of βX.

Thus the question under investigation may be restated:

Is every countably generated z-ideal of $C(X)$ a pure ideal?

1.2. As customary, given a subset $\{f_\alpha\}$ of $C(X)$, (f_α) denotes the ideal generated by that subset; in particular, if $g \in C(X)$, then (g) denotes the principal ideal generated by g.

If $f, g \in C(X)$ and $Z(f)$ is a neighborhood of $Z(g)$ in $C(X)$ then f is a multiple of g in $C(X)$ **[GJ**, 1.0]. Hence, with $\omega =$ the natural numbers, we have the following:

LEMMA. *Let $(f_n)_{n \in \omega}$ be a sequence in $C(X)$ such that* $\operatorname{int}_X(Z(f_n)) \supseteq Z(f_{n+1})$, *for every $n \in \omega$. Then $I = (f_0, f_1, f_2, \ldots)$ is a z-ideal.*

1.3 LEMMA. *Let I be an ideal of $C(X)$. Then*

(i) *I is pure if and only if for each $f \in I$ there exists $g \in I$ such that $X \setminus Z(f)$ and $Z(g)$ are completely separated in X.*

(ii) *If X is normal, then I is pure if and only if for every $f \in I$ there exists $g \in I$ such that* $\operatorname{int}_X Z(f) \supseteq Z(g)$,

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PROOF. (i) By definition of purity (cf. 1.1) I is pure if and only if given $f \in I$ there exists $g \in I$ such that $f = fg$. If I is pure, then $1 - g$ completely separates $X \setminus Z(f)$ and $Z(g)$; conversely, if $f, g \in I$, and $v \in C(X)$ is 1 on $X \setminus Z(f)$ and 0 on a neighborhood of $Z(g)$, then $f = fv$, and $v \in I$, being a multiple of g. Thus I is pure.

(ii) follows obviously from (i) since, in normal spaces, two subsets are completely separated iff they have disjoint closures.

1.4 LEMMA. *Every countably generated semiprime ideal I of $C(X)$ is generated by a sequence $g_0, g_1, g_2, \ldots$ such that*

$$(g_0) \subset (g_1) \subset (g_2) \subset \cdots.$$

PROOF. If $I = (f_0, f_1, f_2, \ldots)$, put $g_n = \sum_{i=0}^{n} |f_i|^{1/2}$ (cf. [$\mathbf{D}_1$, 3]).

When X is first countable, there is a converse to Lemma 1.2:

1.5 LEMMA. *Assume X is first countable. Then every countably generated z-ideal I of $C(X)$ has a sequence of generators $f_0, f_1, f_2, \ldots$ such that* $\operatorname{int}_X Z(f_n) \supset Z(f_{n+1})$, *for every $n \in \mathbf{N}$.*

PROOF. Being a z-ideal, I is semiprime, hence it has a sequence of generators $g_0, g_1, g_2, \ldots$ with $(g_0) \subset (g_1) \subset (g_2) \subset \cdots$ by Lemma 1.4. We prove that $g_0, g_1, g_2, \ldots$ admits a subsequence with the required property. Put $n(0) = 0$. If $\operatorname{int}_X Z(g_0)$ fails to contain any $Z(g_n)$, then for each n there exists $p_n \in \operatorname{cl}_X \operatorname{coz}(g_0) \cap Z(g_n)$ and a sequence $(x_m^n)_m$ in $\operatorname{coz}(g_0)$, which converges to p_n as $m \to \infty$. For each n, the set $S_n = \operatorname{cl}_{\beta X} Z(g_0) \cup \{x_m^n: m \in \mathbf{N}\}$ is then a closed subset of βX; the function which is zero on $\operatorname{cl}_{\beta X} Z(g_0)$ and $|g_n(x_m^n)|^{1/2}$ on $\{x_m^n: m \in \mathbf{N}\}$ is clearly continuous on S_n; hence it has a continuous extension $h_n^* \in C(\beta X)$. Let $h_n = h_n^*|X$. Put $h = \sum_{n=0}^{\infty}(h_n \wedge 2^{-n})$; by uniform convergence, $h \in C(X)$. Clearly $Z(h) \supset Z(g_0)$; but h cannot be a multiple (in $C(X)$) of any g_n. For if $h = ug_k$, then $ug_k > h_k \wedge 2^{-k}$; if m is sufficiently large, we have $h_k(x_m^k) \wedge 2^{-k} = |g_k(x_m^k)|^{1/2}$, hence $|u(x_m^k)| > 2^{-k}|g_k(x_m^k)|^{-1/2}$; letting $m \to \infty$, we see that u cannot be continuous at p_k. This contradiction shows that there exists $n(1) \in \mathbf{N}$ such that $\operatorname{int}_X Z(g_0) \supset Z(g_{n(1)})$. An obvious induction yields the desired subsequence.

1.6 PROPOSITION. *If X is normal and first countable, then every countably generated z-ideal of $C(X)$ is pure. Hence, in such spaces, countably generated z-ideals are the ideals O^A, with A running on the zero-sets of βX.*

PROOF. By Lemmas 1.5 and 1.3(ii).

2. The example. In this section we describe a space X and a countably generated z-ideal I of $C(X)$ that is not pure. (What we construct is a space X and a sequence $(Z_n)_{n \in \omega}$ of zero-sets of X such that $\operatorname{int}_X Z_n \supset Z_{n+1}$, but $X \setminus Z_0$ and Z_n are not completely separated, for every $n \in \omega$; this disproves the conjecture in question (Lemma 3(a) and Lemma 2, §1).)

The space is obtained by attaching together, in a certain way, infinitely many copies of Tychonoff planks; $\omega = \omega_0$ and ω_1 are the cardinals $\aleph_0$, $\aleph_1$, respectively, considered as initial ordinals, with their order topology. Consider the (complete) Tychonoff planks $T^* = (\omega_1 + 1) \times (\omega + 1)$; let S^* denote the space obtained from T^* by identifying all points of the "top edge" $(\omega_1 + 1) \times \{\omega\}$ to a point c. It is straightforward to see that S^* is a compact Hausdorff space with clopen basis. (One can picture it as a set of "rays" centered at c, each ray being a copy of $\omega + 1$, with c as limit point.) The (complete) *book* is the space $B^* = S^* \times (\omega_1 + 1)$, with product topology. Clearly, B^* is a compact Hausdorff space with clopen basis. The subspace $W^* = \{c\} \times (\omega_1 + 1)$ is a copy of $\omega_1 + 1$; it is called the *back* of B^*. Points of $B^* \setminus W^*$ are triples (α, m, β) with $\alpha, \beta \in \omega_1 + 1$, $m \in \omega$; α, m, β may be thought of as "cylindrical coordinates" α being the angle, m the radius, β the height; but it is more fitting to think of α as the page in which the point lies, and m, β as the *column* and the *line*, respectively at which one finds the point on the αth page. Thus, the book has $\aleph_1$ pages, each page has $\omega = \aleph_0$ columns and $\aleph_1$ lines. For a better understanding of the arguments which follow it is useful to keep this picture in mind. The *top section* of B^* is the subspace $S^* \times \{\omega_1\}$; for each $m \in \omega$ we have the mth *top column* $C_m = \{(\alpha, m, \omega_1): \alpha \in \omega_1 + 1\}$ it is a copy of $\omega_1 + 1$. The *vertex* of B^* is the point $v = (c, \omega_1)$ $(\in W^*)$. The *top edge* E of B^* is the subspace $\{(\omega_1, m, \omega_1): m \in \omega\} \cup \{v\}$; it is a copy of $\omega + 1$. The *incomplete book* B is the subspace $B^* \setminus E$.

We need some facts on B, B^*.

2.1 LEMMA. (i) *W^* is a zero-set of B^*.*

(ii) *B is an open dense C-embedded subspace of B^*. (Hence, B is pseudocompact, and $B^* = \beta B$.)*

PROOF. (i) Define $\phi: B^* \to \mathbf{R}$ to be 0 on W^*, and put $\phi(\alpha, m, \beta) = 2^{-m}$ for every $(\alpha, m, \beta) \in B^* \setminus W^*$. It is easy to see that ϕ is continuous. The proof of (ii) is deferred until §3. □

Consider now the space $\Lambda^* = (\omega \times \omega) \times B^*$, with product topology. Since $\omega \times \omega$ is discrete, Λ^* is simply a topological sum of $\aleph_0$ disjoint copies of B^*, it is locally compact and σ-compact (hence realcompact) but not compact. It is called the (complete) *library*; its subspace $\Lambda = (\omega \times \omega) \times B$, the *incomplete library*, is open dense and C-embedded in Λ^* (Lemma 2.1(ii)); then Λ^* is the realcompactification of Λ. Given $r \in \omega$, the subspace $\Sigma_r^* = (\{r\} \times \omega) \times B^*$ is called the (complete) rth *shelf* of Λ^*; and $(\{r\} \times \{s\}) \times B^*$ is the sth book of the rth shelf (the meaning of Σ_r, incomplete rth shelf, should be obvious). We now "attach" to each other the books of Λ^* in a certain way; we shall obtain a quotient X^* of Λ^*, which will have the image X of Λ as dense and C-embedded subspace; the space X will yield our example. Fix a bijection $u: \omega \to \omega \times \omega$, $u(s) = (s_1, s_2)$. Define a (noncontinuous) map $\mu: \Lambda^* \to \Lambda^*$ piecewise, as follows: points of Λ^* which do not belong to the back of any book are left fixed; if p lies in the back of the sth book of the rth shelf, say $p = ((r, s), (c, \beta))$, $\beta \in \omega_1 + 1$, then define $u(p) = ((r + 1, s_1), (\beta, s_2, \omega_1))$. In other words, μ "attaches" the back of the sth book of the rth shelf (this back is a copy of $\omega_1 + 1$) to the s_2th top column of the s_1th book of the $(r + 1)$st shelf. (This column is a copy of $\omega_1 + 1$, too.) The equivalence relation identifies x and $\mu(x)$, for every $x \in \Lambda^*$; since μ is idempotent (i.e. $\mu \circ \mu = \mu$, as is easy to see) each equivalence class contains at most two points (exactly one if the point does not lie on the back or on the "top section" $S^* \times \{\omega_1\}$ of any book); this implies that the quotient is a T_1-space (but we do not need, at this point, any separation property for the quotient; in §3 it shall be proved that the quotient is a T_4 zero-dimensional space).

Let X^* be the quotient, q: $\Lambda^* \to X^*$ the quotient map. It is easy to verify that $q^{\leftarrow}(q(\Lambda)) = \Lambda$ (vertices of books are mapped by μ into points of top edges); since Λ is open (and dense) in Λ^*, this implies that $X = q(\Lambda)$ is open (and dense) in X^*, moreover the quotient topology of X (*via* the map $q|\Lambda$) coincides with the subspace topology.

By well-known properties of quotients, $f \in C(\Lambda^*)$ may be factored as $g \circ q$, with $g \in C(X^*)$ if and only if f is constant on the equivalence classes of q; this may be restated as $f \circ \mu = f$. The same holds true for Λ and X, of course. We claim that

2.2 LEMMA. *X is dense and is C-embedded in X^*. Hence $C(X)$ and $C(X^*)$ are isomorphic.*

PROOF. Given $g \in C(X)$, consider $h = g \circ (q|\Lambda) \in C(\Lambda)$. As observed before, Λ is dense and C-embedded in Λ^*; let h^* denote the continuous extension of h on Λ^*. Since $h \circ \mu = h$, also $h^* \circ \mu = h^*$ (given a vertex of some book in Σ_r^*, μ maps it into some point of the top edge of some book of Σ_{r+1}^*, h must eventually assume the same value on tails of the back of the book in the rth shelf and in the top column of the book of the $(r + 1)$st shelf to which μ attaches this back). Hence h^* factors as $g^* \circ q$, with $g^* \in C(X^*)$ being the required extension of g.

We now define a sequence $(f_n)_{n\in\omega}$ of functions of $C(\Lambda^*)$, compatible with the equivalence relation; the sequence $(g_n^*)_{n\in\omega}$ of $C(X^*)$ (where $g_n^* \circ q = f_n$ for each $n \in \omega$) will be shown to generate a z-ideal of $C(X^*)$ which is not pure. Define f_n to be identically zero on $\bigcup_{r>n+1} \Sigma_r^*$. This forces us to put f_n identically zero on all the backs of books of Σ_n^*; we define f_n on the other points of books of the nth shelf by means of the function ϕ used in the proof of Lemma 1(i) of this section. On the mth top columns of books in the nth shelf, f_n has then constant value 2^{-m}; use this constant value to define f_n on books of the $(n - 1)$st shelf attached to these columns. Repeating this last procedure, it is easy to define f_n (piecewise constant) on all books of all shelves of lower degree. It is clear that f_n is q-compatible, i.e. $f_n = g_n^* \circ q$ with $g_n^* \in C(X^*)$. Put $g_n = g_n^*|X$.

Denote by I the ideal generated by $(g_n)_{n\in\omega}$ in $C(X)$, by I^* the (isomorphic) ideal in the (isomorphic) ring $C(X^*)$.

2.3 LEMMA. (i) *I is a z-ideal of $C(X)$.*
(ii) *I^* is not pure in $C(X^*)$.*

PROOF (i) We prove that $\operatorname{int}_X Z_X(g_n) \supseteq Z(g_{n+1})$, for every $n \in \omega$ [cf. Lemma 1.3]. In fact $q^{\leftarrow}(Z_X(g_{n+1})) \subseteq A \subseteq q^{\leftarrow}(Z_X(g_n))$ where A is $\bigcup_{r>n+1} \Sigma_r$ without the point of the "top sections" of the books of Σ_{n+1}, i.e. $A = (\bigcup_{r>n+1} \Sigma_r) \setminus q^{\leftarrow}q\, \Sigma_n$. Since A is an open subset of Λ and is $A = q^{\leftarrow}q(A)$, the required result follows.

(ii) We prove that $A = X^* \setminus Z_{X^*}(g_0^*)$ and $Z_{X^*}(g_n^*)$ are not completely separated, for any $n \in \omega$. In fact, the closure of A in X^* meets $Z_{X^*}(g_n^*)$, as we now show. Take any vertex of any book of the rth shelf, for any $r \geqslant n$, say $((r, s), v)$. Then $p = q((r, s), v) \in Z_{X^*}(g_n^*)$; and any open neighbourhood V^* of p in X^* meets A. For, $q^{\leftarrow}(V^*)$ is an open neighbourhood of $((r, s), v)$ in Λ^*; then $q^{\leftarrow}(V^*)$ contains infinitely many top columns of the sth book of the rth shelf; by equivalence, $q^{\leftarrow}(V^*)$ contains also the backs of infinitely many books of the $(r-1)$st shelf; it is then a neighbourhood of infinitely many vertices of books in this shelf, repeating the procedure, we see that $q^{\leftarrow}(V^*)$ is a neighbourhood of infinitely many vertices of books of Σ_0^*; thus $q^{\leftarrow}(V^*) \cap \mathrm{Coz}_{\Lambda^*}(f_0) \neq \varnothing$; hence V^* meets $A = q(\mathrm{Coz}_{\Lambda^*}(f_0))$.

3. In this section we give a proof of (ii) of Lemma 2.1 which asserts that B is C-embedded in B^*.

We shall also prove that X and X^* are Hausdorff spaces with clopen basis.

PROOF. Let $f \in C(B)$. We first prove that f extends to a continuous function (which we still call f) on $B^* \setminus \{v\}$. Observe that for every $m \in \omega$ the subspace $C_m^* = \{(\alpha, m, \beta)\colon \alpha, \beta \in \omega_1\}$ is clopen in B^* and is homeomorphic to the space $\Omega^* = (\omega_1 + 1) \times (\omega_1 + 1)$ of [GJ, 8L]. From the same reference, we know that $\Omega = \Omega^* \setminus \{(\omega_1, \omega_1)\}$ is C-embedded in Ω^*. Clearly $C_m = C_m^* \cap B$ is a copy of Ω.

Given $f \in C(B^* \setminus \{v\})$, observe that f is eventually constant on the back, i.e. there exists $\bar\beta \in \omega_1$ such that $f(c, \beta) = f(c, \bar\beta)$ for all $\beta \in \omega_1$, $\beta \geqslant \bar\beta$. For simplicity, assume this eventual value to be 0; and put $f(v) = 0$. The extension so obtained is continuous: otherwise, there exist $\varepsilon > 0$ and a sequence $p_n = (\alpha_n, m_n, \beta_n)$, with $\beta_n \geqslant \bar\beta$, $\beta_n \in \omega_1$, $\alpha_n \in \omega_1 + 1$, and $m_n \in \omega$, $\lim_{n\to\infty} m_n = \infty$, such that $|f(p_n)| \geqslant \varepsilon$. In the compact space B^*, $(p_n)_n$ has cluster points; since $m_n \to \infty$ as $n \to \infty$, such a cluster point belongs to the back W^*; and since $\bar\beta \leqslant \sup_n \beta_n < \omega_1$, such a cluster point is necessarily some (c, β'), with $\bar\beta < \beta' \leqslant \omega_1$. Then $f(c, \beta') = 0$; but continuity of f at (c, β') implies $|f(c, \beta')| \geqslant \varepsilon$, a contradiction. This ends the proof.

Given a complete book B^*, and a point $(\bar\alpha, m, \bar\beta) \in B^* \setminus W^*$ a neighbourhood base for it in B^* consists of "rectangles"

(1) $V_{\alpha,\beta} = \{(\xi, m, \eta) \in B^*\colon \alpha < \xi \leqslant \bar\alpha, \beta < \eta \leqslant \bar\beta\}$ with $\alpha < \bar\alpha$, $\beta < \bar\beta$.

For a point $(c, \bar\beta) \in W^*$ a neighbourhood base is

(2) $V_{m,\beta} = \{(c, \eta)\colon \beta < \eta \leqslant \bar\beta\} \cup \{(\xi, n, \eta)\colon \xi \in \omega_1 + 1,\ n > m;\ \beta < \eta \leqslant \bar\beta\}$ with $m \in \omega$, $\beta < \bar\beta$.

Next, a neighbourhood base for the back W^* of B^* consists of the sets

(3) $V_m = W^* \cup \{(\alpha, n, \beta)\colon \alpha, \beta \in \omega_1 + 1, n > m\}$, with $m \in \omega$.

All these sets are obviously clopen in B^*.

3.1 PROPOSITION. *X^* has a clopen basis.*

PROOF. Take a point $p \in \Lambda^*$, p belongs to some book of, say, the nth shelf Σ_n^*. If p is neither in the back, nor in the top section (see §2) of the book, then there exists a neighbourhood $V_{\alpha,\beta}$ of p in the book (type 1) such that the equivalence relation is the identity on $V_{\alpha,\beta}$. Then $q^{\leftarrow}(q(V_{\alpha,\beta})) = V_{\alpha,\beta}$, and $q(p)$ has a clopen neighbourhood basis in X^*. Assume now that $p = \mu(p')$, with p' in the back of some book in Σ_{n-1}^*. Take for p' a clopen neighbourhood V_1 of type 2, $V_{m,\beta'}$; take for p, in its book a neighbourhood V_0 (type 1) $V_{\beta',\beta}$. Assuming that p' is not the vertex of its book, we have $V_0 \cup V_1 = q^{\leftarrow}q(V_0 \cup V_1)$, so that $q(p)$ has a clopen neighbourhood basis in X^*. Finally assume $p = \mu(p')$, with $p' = v$, vertex of some book in Σ^*_{n-1}. Then the books of Σ^*_{n-1} are either disjoint from $\mu^{\leftarrow}(V_1)$, or, otherwise, their back is contained in $\mu^{\leftarrow}(V_1)$. Choose a clopen neighbourhood of these backs of type 3, and consider their union V_2. Repeating this procedure with V_2 in place of V_1, and so on inductively, we obtain a clopen subset of Λ invariant under μ and $\mu^{\leftarrow}$; and the proof is complete.

3.2 PROPOSITION. *X^* is a σ-compact Hausdorff space with clopen basis. Moreover, X^* is the realcompactification of X.*

PROOF. As was observed in §2, X^* is a T_1-space. Since Λ^* is σ-compact, so is X^*. The remaining assertions follow easily.

OBSERVATION. It is not difficult to show that X is locally compact (but not σ-compact, being not even realcompact). And X^* is σ-compact, but not locally compact.

ADDED IN PROOF. The conjecture is true also for locally compact normal spaces. The proof, quite similar to the first countable case, will appear in a forthcoming paper.

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