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BEWARE THE PHONY MULTIPLICATION ON QUILLEN'S $\mathcal{A}^{-1}\mathcal{A}$

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ABSTRACT. This paper exposes a subtle but irreparable flaw in a frequently proposed construction of the graded ring structure on the algebraic K-groups of commutative rings.

The purpose of this note is to warn the reader away from an apparently clear and simple procedure for defining a pairing $K_*(A) \otimes K_*(B) \to K_*(A \otimes B)$ of the algebraic K-groups of rings. A convincing construction for doing this involving Quillen's $\mathcal{A}^{-1}\mathcal{A}$ construction [3] has occurred independently to various mathematicians, including the author. It has been presented in seminars at Chicago and Princeton, and has appeared in print [2]. Nevertheless, this construction relies upon a subtle but irreparable error, and is totally fallacious.

There are other, valid methods of constructing pairings of algebriac K-groups, due to Loday [4], Waldhausen [11, §9], Segal and Wolfson [9], [12], and May [6, VIII], [8]. These methods are unfortunately more complicated. Further, May's first attempt at the pairings [6] suffers from a related error which is fixed in [8], and Segal and Wolfson barely evade the analogous problem in their work.

The mistaken construction of the pairing is expounded exceptionally lucidly in [2], whose treatment I follow. Accordingly, I will start with some generalities.

Recall that a symmetric monoidal category is a category $\mathcal{A}$ provided with a distinguished object 0 and a functor $\oplus$: $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$. There are to be natural isomorphisms

$$\begin{gathered}(A \oplus B) \oplus C \cong A \oplus (B \oplus C),\\ A \oplus B \cong B \oplus A, \quad A \oplus 0 \cong A. \end{gathered} \tag{1}$$

These isomorphisms are required to satisfy certain "coherence" conditions. More or less exhaustive treatments may be found in [1, III, §1; II, §1], [5, VII, §§1, 7], [7, §4]. One speaks of $\oplus$ as a sort of sum. The example of most interest to algebraic K-theory is to take R a ring, and let $\mathcal{A}$ be the category whose objects are fintely generated projective R-modules and whose morphisms are R-linear isomorphisms. Here $\oplus$ is given by direct sum.

Such a symmetric monoidal category $\mathcal{A}$ has a Quillen-Grothendieck completion, $\mathcal{A}^{-1}\mathcal{A}$. This is a symmetric monoidal category whose objects are pairs (A, B) of objects of $\mathcal{A}$. A morphism $(A, B) \to (C, D)$ in $\mathcal{A}^{-1}\mathcal{A}$ is an equivalence class of data consisting of an object S of $\mathcal{A}$ and two morphisms in $\mathcal{A}$, α: $A \oplus S \to C$, β: $B \oplus C \to D$. The datum $(S; \alpha, \beta)$ is equivalent to $(S'; \alpha', \beta')$ if there is an

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isomorphism $\gamma: S \xrightarrow{\sim} S'$ such that the diagrams (2) commute.

$$\begin{tikzcd} A \oplus S \arrow[dd, "A \oplus \gamma"', "\wr"] \arrow[dr, "\alpha"] & \\ & C \\ A \oplus S' \arrow[ur, "\alpha'"'] & \end{tikzcd} \qquad \begin{tikzcd} B \oplus S \arrow[dd, "B \oplus \gamma"', "\wr"] \arrow[dr, "\beta"] & \\ & D \\ B \oplus S' \arrow[ur, "\beta'"'] & \end{tikzcd} \tag{2}$$

Given morphisms $(A, B) \to (C, D)$ and $(C, D) \to (E, F)$ represented by data $(S; \alpha, \beta)$, $(T; \gamma, \delta)$ respectively, the composite $(A, B) \to (E, F)$ is represented by the datum consisting of $S \oplus T$ and the two morphisms $\gamma \cdot \alpha \oplus T$, $\delta \cdot \beta \oplus T$.

There is an inclusion functor i: $\mathcal{A} \to \mathcal{A}^{-1}\mathcal{A}$ given on objects by $i(A) = (0, A)$ and on morphisms by $i(\alpha) = (0; 1, \alpha)$.

Suppose now and henceforth that $\mathcal{A}$ satisfies the conditions that every morphism is an isomorphism and that, for every object A of $\mathcal{A}$, the functor $A \oplus \_ : \mathcal{A} \to \mathcal{A}$ is faithful. Then i: $\mathcal{A} \to \mathcal{A}^{-1}\mathcal{A}$ induces a group completion of classifying spaces Bi: $B\mathcal{A} \to B\mathcal{A}^{-1}\mathcal{A}$. That is, $\pi_0 B\mathcal{A}^{-1}\mathcal{A}$ is the Grothendieck group of the monoid $\pi_0 B\mathcal{A}$, and the homology of $B\mathcal{A}^{-1}\mathcal{A}$ is the localization of the homology of $B\mathcal{A}$ with respect to the action of $\pi_0 B\mathcal{A}$. Thus, for $\mathcal{A}$ the category of finitely generated projective R-modules above, $K_i(R) = \pi_i B\mathcal{A}^{-1}\mathcal{A}$. For a full discussion, see **[3]**.

$B\mathcal{A}^{-1}\mathcal{A}$ is thus an H-space with multiplication induced by the obvious symmetric monoidal structure on $\mathcal{A}^{-1}\mathcal{A}$, and it has a homotopy inverse as $\pi_0 B\mathcal{A}^{-1}\mathcal{A}$ is a group. One may regard an object (B, A) of $\mathcal{A}^{-1}\mathcal{A}$ as the virtual object $A - B$ of $\mathcal{A}$, as it represents this class in $\pi_0 B\mathcal{A}^{-1}\mathcal{A}$. There is a functor ι: $\mathcal{A}^{-1}\mathcal{A} \to \mathcal{A}^{-1}\mathcal{A}$ given on objects by $\iota(A, B) = (B, A)$ and on morphisms by $\iota(S; \alpha, \beta) = (S; \beta, \alpha)$. It seems plausible that the map $B\iota$ is the homotopy inverse for the H-space structure on $B\mathcal{A}^{-1}\mathcal{A}$, as it induces the inverse on the group $\pi_0 B\mathcal{A}^{-1}\mathcal{A}$. Indeed, this is so **[10]**, but the argument usually given to prove this is wrong. The usual argument is as follows: Let 0 denote the constant functor sending everything to $(0, 0)$. Suppose there were a natural transformation η: $0 \to \iota \oplus \mathrm{Id}$. Then the induced maps of classifying spaces, $B0$ and $B(\iota \oplus \mathrm{Id})$ would be homotopic. As $B0$ sends everything to the basepoint, and $B(\iota \oplus \mathrm{Id})$ represents the homotopy class of the H-space sum of the maps $B\iota$ and $B(\mathrm{Id})$, this would imply that $B\iota = -B\,\mathrm{Id}$ is the homotopy inverse for the H-space structure. This much of the argument is valid. The error comes in asserting the existence of such a natural transformation η.

There is an obvious candidate for η. The functor $\iota \oplus \mathrm{Id}$: $\mathcal{A}^{-1}\mathcal{A} \to \mathcal{A}^{-1}\mathcal{A}$ sends the object (A, B) to $(B \oplus A, A \oplus B)$. So to give a natural transformation η, one must give a morphism $\eta(A, B)$: $(0, 0) \to (B \oplus A, A \oplus B)$ for each (A, B) and check naturality. The obvious choice for $\eta(A, B)$ is the morphism determined by the datum $(A \oplus B, \tau, 1)$ where τ: $0 \oplus A \oplus B \simeq B \oplus A$ and 1: $0 \oplus A \oplus B \simeq A \oplus B$ are the canonical isomorphisms. However, these $\eta(A, B)$ do not determine a natural transformation.

Let $(S; \alpha, \beta)$: $(A, B) \to (C, D)$ be a morphism in $\mathcal{A}^{-1}\mathcal{A}$. Then $\iota \oplus \mathrm{Id}$ sends this to the morphism $(B \oplus A, A \oplus B) \to (D \oplus C, C \oplus D)$ determined by the datum consisting of the object $S \oplus S$, and the two morphisms (3).

$$\begin{aligned} &B \oplus A \oplus S \oplus S \overset{\sigma}{\cong} B \oplus S \oplus A \oplus S \underset{\sim}{\overset{\beta\oplus\alpha}{\to}} D \oplus C, \\ &A \oplus B \oplus S \oplus S \overset{\sigma'}{\cong} A \oplus S \oplus B \oplus S \underset{\sim}{\overset{\alpha\oplus\beta}{\to}} C \oplus D. \end{aligned} \tag{3}$$

Here σ and σ' are the canonical isomorphisms that permute the factors but do not reverse the relative position of the two copies of S. (Note I have suppressed any mention of ways of inserting parentheses in the "sums" and of the natural associativity isomorphisms: these are irrelevant here and one may assume $\mathcal{A}$ is permutative if one likes.) The condition that η is a natural transformation is just that for all such $(S; \alpha, \beta)$, that $\eta(C, D) = \iota \oplus \mathrm{Id}(S; \alpha, \beta) \cdot \eta(A, B)$. The composite $\iota \oplus \mathrm{Id}(S; \alpha, \beta) \cdot \eta(A, B)$ is the morphism determined by the datum $(A \oplus B \oplus S \oplus S; \mu, \nu)$, where the morphisms μ, ν are the composite (4).

$$\begin{aligned} &\mu\colon 0 \oplus A \oplus B \oplus S \oplus S \underset{\sim}{\overset{\tau\oplus S\oplus S}{\to}} B \oplus A \oplus S \oplus S \underset{\sim}{\overset{\sigma}{\to}} B \oplus S \oplus A \oplus S \underset{\sim}{\overset{\beta\oplus\alpha}{\to}} D \oplus C, \\ &\nu\colon 0 \oplus A \oplus B \oplus S \oplus S \underset{\sim}{\overset{1}{\to}} A \oplus B \oplus S \oplus S \underset{\sim}{\overset{\sigma}{\to}} A \oplus S \oplus B \oplus S \underset{\sim}{\overset{\alpha\oplus\beta}{\to}} C \oplus D. \end{aligned} \tag{4}$$

On the other hand, $\eta(C, D)$ is determined by the datum $(C \oplus D; \tau', 1)$. For these two data to be equivalent, and so to determine the same morphism, there must be an isomorphism $\gamma\colon A \oplus B \oplus S \oplus S \to C \oplus D$ such that $\mu = \tau' \cdot \gamma$ and $\nu = 1 \cdot \gamma$. Thus $\nu = \gamma$, and the condition to be satisfied is that $\mu = \tau' \cdot \nu$ in $\mathcal{A}$. This condition is equivalent to the condition that the diagram (5) commutes.

$$\begin{array}{ccccccc} B \oplus A \oplus S \oplus S & \underset{\sim}{\overset{\sigma}{\to}} & B \oplus S \oplus A \oplus S & = & (B \oplus S) & \oplus & (A \oplus S) \\ {\scriptstyle \tau\oplus S\oplus S}\uparrow\wr & & & & & \wr\uparrow{\scriptstyle \tau'} & \\ A \oplus B \oplus S \oplus S & \underset{\sim}{\overset{\sigma}{\to}} & A \oplus S \oplus B \oplus S & = & (A \oplus S) & \oplus & (B \oplus S) \end{array} \tag{5}$$

Here all morphisms are the canonical isomorphisms that permute factors in the "sums". If one goes from the lower left-hand corner to the upper right-hand corner via the top and left side of the diagram, the isomorphism does not transpose the two copies of S, as σ does not. On the other hand, if one follows the bottom and right side of the diagram, the isomorphism does transpose the two copies of S, as τ' does. Thus the above diagram is the sum of a commutative diagram and the diagram (6), in which τ' permutes the two factors.

$$\begin{array}{ccc} S \oplus S & = & S \oplus S \\ \| & & \wr\uparrow{\scriptstyle \tau'} \\ S \oplus S & = & S \oplus S \end{array} \tag{6}$$

Thus the diagram always commutes and so η is natural, if and only if for all S, the isomorphism $\tau'\colon S \oplus S \cong S \oplus S$ permuting the two factors is also the identity map. This condition is clearly not met by most $\mathcal{A}$. For $\mathcal{A}$ the category of finitely generated projective R-modules, it holds only if R is the degenerate ring consisting of 0 alone. Thus for $R \neq 0$, η is not natural for such $\mathcal{A}$.

I note that η was claimed to be natural in the preprint version of [3], but this is not essential to the arguments there.

Once it is realized that η is not a natural transformation, all the results of [2] are invalidated. There it is claimed [2, Proposition 1] that $\mathcal{A}^{-1}\mathcal{A}$ has a universal mapping property characterizing it among symmetric monoidal categories that possess functors ι and natural transformations η: $0 \to \iota \oplus \mathrm{Id}$ satisfying certain axioms. As $\mathcal{A}^{-1}\mathcal{A}$ does not carry this sort of structure itself, this proposition is false. But this proposition was the key to constructing the pairings.

Suppose $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ are three symmetric monoidal categories and that there is a functor $\otimes$: $\mathcal{A} \times \mathcal{B} \to \mathcal{C}$ which is a pairing in that the appropriate "distributive laws" of $\otimes$ over "sums" $\oplus$ hold up to coherent natural isomorphism. For example, $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ could be the categories of finitely generated projective modules over rings R, S, and $R \otimes S$, with $\otimes$ given by tensor product over $\mathbf{Z}$.

One would like to see the corresponding pairing of algebraic K-groups induced on homotopy groups by a map $B\mathcal{A}^{-1}\mathcal{A} \wedge B\mathcal{B}^{-1}\mathcal{B} \to B\mathcal{C}^{-1}\mathcal{C}$. To obtain this map, one wishes to start with a functor $\mathcal{A}^{-1}\mathcal{A} \times \mathcal{B}^{-1}\mathcal{B} \to \mathcal{C}^{-1}\mathcal{C}$ induced from the functor $\otimes$: $\mathcal{A} \times \mathcal{B} \to \mathcal{C}$. In [2], this functor is built from the functor $\otimes$ by repeated use of the purported universal mapping property of the $\mathcal{A}^{-1}\mathcal{A}$ construction. But as this universal mapping property is not possessed by $\mathcal{A}^{-1}\mathcal{A}$, the construction of the pairing in [2] is invalid.

Most mathematicians who have claimed to have constructed the phony map $\mathcal{A}^{-1}\mathcal{A} \times \mathcal{B}^{-1}\mathcal{B} \to \mathcal{C}^{-1}\mathcal{C}$ have done so by giving explicit formulae. This approach is more clumsy than the clever construction in [2], and is invalid for similar reasons. I conclude by exposing this error.

For objects (A', A) in $\mathcal{A}^{-1}\mathcal{A}$ and (B', B) in $\mathcal{B}^{-1}\mathcal{B}$, the pairing $\mathcal{A}^{-1}\mathcal{A} \times \mathcal{B}^{-1}\mathcal{B} \to \mathcal{C}^{-1}\mathcal{C}$ should have

$$(A', A) \otimes (B', B) = (A' \otimes B \oplus A \otimes B', A' \otimes B' \oplus A \otimes B).$$

This is clear if one thinks of (A', A) as A minus A', etc. There are a few obvious possibilities for what $(S; \alpha, \beta) \otimes (T; \gamma, \delta)$ should be for morphisms $(S; \alpha, \beta)$ in $\mathcal{A}^{-1}\mathcal{A}$ and $(T; \gamma, \delta)$ in $\mathcal{B}^{-1}\mathcal{B}$. The possible choices differ very slightly from each other, and all choices fail to produce a functor $\otimes$: $\mathcal{A}^{-1}\mathcal{A} \times \mathcal{B}^{-1}\mathcal{B} \to \mathcal{C}^{-1}\mathcal{C}$, as the formula for $\otimes$ will not preserve composition of morphisms. The reader may try this if he wishes; things become quite tedious to check. The main obstruction is that the automorphism of $(S \otimes T) \oplus (S \otimes T)$ that permutes the two "summands" is not the identity. The problem is very similar to that which prevents η from being a natural transformation.

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