ON FINITELY GENERATED AND PROJECTIVE
EXTENSIONS OF BANACH ALGEBRAS

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ABSTRACT. We show that a finitely generated projective extension B of a commutative complex unitary Banach algebra A induces an open mapping π between the carrier spaces. We next prove that if π is a local homeomorphism then B contains an inertial subalgebra. Finally we present a necessary and sufficient condition for B to be uniform if A is.

1. Introduction. Throughout this paper Banach algebra means a commutative Banach algebra over the complex field $\mathbf{C}$ with an identity element 1. If A is a Banach algebra, then its carrier space, endowed with the Gelfand topology, is denoted by $M(A)$ and its Shilov boundary by $\partial(A)$. For each $a \in A$ we denote by $\hat{a}$ the Gelfand transform of a. The Jacobson radical of A is written $R(A)$.

From now on B will denote a finitely generated projective extension of a fixed Banach algebra A. We suppose B to be endowed with Magid's norm under which B is also a Banach algebra [**10**, Theorem 4, p. 138]. We call π the projection of $M(B)$ onto $M(A)$ induced by the inclusion of A into B. The above situation is illustrated by the following example: take $B = A[x]/(\alpha(x))$ where $\alpha(x)$ is a monic polynomial in $A[x]$ (the so-called Arens-Hoffman extensions of A). For such extensions, the structure of the projection π as well as the study of the properties that B can inherit from A have been the main subject in a series of papers ([**4**], [**7**], [**8**]). The purpose of this paper is to establish some results along these lines for the more general situation.

To state the main results it is convenient to introduce a definition.

If $\psi \in M(B)$, $\phi = \pi(\psi)$, and $m_\phi = \text{Ker } \phi$, then $B/m_\phi B$ is a finite dimensional $\mathbf{C}$-algebra whose carrier space is naturally homeomorphic to $\pi^{-1}(\phi)$. Then we define the multiplicity $m(\psi)$ of ψ as the complex dimension of $e(B/m_\phi B)$, where e is the idempotent element of $B/m_\phi B$ such that the support of $\hat{e}$ is $\{\psi\}$.

THEOREM 1. *If $\psi_1 \in M(B)$ and $\pi^{-1}(\pi(\psi_1)) = \{\psi_1, \ldots, \psi_m\}$, then given any neighborhood W of ψ_1, there exist mutually disjoint neighborhoods V_i of ψ_i and a neighborhood U of $\pi(\psi_1)$ such that*:
   (a) $\pi(V_i) = U$ *for all* i,
   (b) $\pi^{-1}(U) = \bigcup_{i=1}^m V_i$,
   (c) $V_1 \subset W$,

(d) $m(\psi_i) = \sum_{\theta \in \pi^{-1}(\phi) \cap V_i} m(\theta)$, $\phi \in U, i = 1, \ldots, m$.
In particular π is open.

THEOREM 2. *If π is a local homeomorphism, then B contains an A-separable subalgebra B_0 such that $B = B_0 + R(B)$.*

THEOREM 3. *Suppose A is a uniform algebra. Then the following are equivalent:*
(a) *B is a uniform algebra.*
(b) *$m(\psi) = 1$ for each $\psi \in \partial(B)$.*

Theorem 1 and Theorem 3 generalize analogous results for the Arens-Hoffman case in [7] and [4], respectively. The subalgebras whose existence is established in Theorem 2 are usually called inertial subalgebras. Other existence theorems for inertial subalgebras have been found by Ingraham in [6] in a more general context.

In the following sections we proceed to prove the theorems stated above. For basic facts and terminology concerning algebras over commutative rings, (algebraically) separable algebras and projective modules the reader is referred to [3].

2. Proof of Theorem 1.[1] If $\alpha(x) = \sum_{i=0}^{n} \alpha_i x^i \in A[x]$ and $\phi \in M(A)$ we set $\alpha_\phi(x) = \sum_{i=0}^{n} \phi(\alpha_i) x^i$ and $Z(\alpha_\phi) = \{\lambda \in \mathbf{C} / \alpha_\phi(\lambda) = 0\}$. For $\lambda \in Z(\alpha_\phi)$, we let $M(\lambda, \alpha_\phi)$ stand for the multiplicity of λ as a root of α_ϕ. If $\alpha(x)$ is monic we write A_α for the Arens-Hoffman extension $A[x]/(\alpha(x))$ and π_α for the projection of $M(A_\alpha)$ onto $M(A)$. We recall that a finitely generated projective module M is said to have a well-defined rank n if for any prime ideal p of A the localized module M_p (which is free) has rank n over the local ring A_p. In this case, we have $\dim_{\mathbf{C}} M/m_\phi M = n$ for each $\phi \in M(A)$.

LEMMA. *Suppose that B has well defined rank, say n. Then for each $b \in B$ there is a monic polynomial $\alpha(x) \in A[x]$, of degree n, and a continuous mapping $f^*: M(B) \to M(A_\alpha)$, commuting with projections, such that:*
(a) *f^* is onto.*
(b) *$M(\psi(b), \alpha_{\pi(\psi)}) = \sum m(\theta)$, $\psi \in M(B)$,*
the sum being taken over those $\theta \in M(B)$ satisfying $f^*(\theta) = f^*(\psi)$.

PROOF. Fixed $b \in B$, we consider the A-endomorphism u of B defined by $u(c) = cb$, $c \in B$. Let $\alpha(x) = \det(xI - u)$ be the characteristic polynomial of u (see [1, exercise 9 (c), p. 148] for a definition). We have $\alpha(b) = 0$ and $n = $ degree $\alpha(x)$. Let f be the A-algebra homomorphism from A_α into B determined by the condition $f([x]) = b$, where $[x]$ denotes the class of x in A_α. Let f^* be the dual map of f. In order to prove (a) we note that a character in $M(A_\alpha)$ is given by a pair (ϕ, λ) with $\phi \in M(A)$ and $\lambda \in Z(\alpha_\phi)$, acting in the following way

$$\sum_{i=0}^{n-1} a_i [x]^i \to \sum_{i=0}^{n-1} \phi(a_i) \lambda^i, \qquad a_i \in A \text{ for each } i.$$

Thus it follows that $f^*(\psi) = (\pi(\psi), \psi(b))$.

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[1] This proof is not the original proposed by the author, which was longer. I am indebted to the referee for the present proof as well as for the statement he suggested of the lemma in this section.

We write u_ϕ for the C-endomorphism of $B/m_\phi B$ obtained from u by factoring by the ideal $m_\phi B$. For each $\psi_i \in \pi^{-1}(\phi)$, let e_i be the idempotent of $B/m_\phi B$ such that the support of $\hat{e}_i$ is $\{\psi_i\}$, $1 \leqslant i \leqslant$ card $\pi^{-1}(\phi)$. Then $B/m_\phi B$ is the direct sum of the subalgebras $e_i(B/m_\phi B)$, which are invariant under u_ϕ. It is easily seen that the unique eigenvalue of the restriction of u_ϕ to $e_i(B/m_\phi B)$ is $\psi_i(b)$, and thus

$$\alpha_\phi(x) = \prod_i (x - \psi_i(b))^{m(\psi_i)}, \qquad \phi \in M(A). \tag{1}$$

Since the character in $M(A_\alpha)$ determined by the pair (ϕ, λ) is in Im f^* if and only if $\lambda = \psi_i(b)$ for some $\psi_i \in \pi^{-1}(\phi)$, (1) proves (a).

To prove (b) it is enough to observe that for ψ, $\theta \in M(B)$, $f^*(\psi) = f^*(\theta)$ is equivalent to $\psi(b) = \theta(b)$ and $\pi(\psi) = \pi(\theta)$.

Now we can proceed to prove Theorem 1. By [3, 4.11, p. 31], there are mutually orthogonal idempotents $e_1, \ldots, e_p$ of A, such that $e_1 + \cdots + e_p = 1$ and $e_i B$ is a finitely generated projective extension of $e_i A$ with a well defined rank (over $e_i A$) for each i. Hence without loss of generality we may assume that B has a well-defined rank over A, say n.

If we put $B_0 = B \otimes_A C(M(A))$ (we denote by $C(X)$ the algebra of continuous complex functions over a compact space X), then B_0 is a finitely generated projective extension of $C(M(A))$ [3, 2.1, p. 12] with a well-defined rank over $C(M(A))$ (in fact, the rank is n). It can easily be seen that $M(B_0)$ is homeomorphic to $M(B)$ under the map h induced by the A-algebra homomorphism $b \to b \otimes 1$ of B into B_0. But h preserves the multiplicity, so we may also assume that $A = C(M(A))$.

Let $b_1, \ldots, b_k$ be a set of generators of B over A and let $\alpha_1, \ldots, \alpha_k$ be monic polynomials in $A[x]$ satisfied by $b_1, \ldots, b_k$ respectively. Put $B_1 = ((\cdots(A_{\alpha_1})_{\alpha_2})\cdots)_{\alpha_k}$, the iterated Arens-Hoffman extension generated by $\alpha_1, \ldots, \alpha_k$. By construction we have a continuous A-algebra homomorphism from B_1 onto B. Applying k times [8, 4.2, p. 564] we obtain that $\hat{B}_1$ is uniformly dense in $C(M(B_1))$. From this it follows that $\hat{B}$ is uniformly dense in $C(M(B))$.

Using this fact, we can consider $b \in B$ and $\varepsilon > 0$ such that $\hat{b}$ separates the points of $\pi^{-1}(\pi(\psi_1))$ and

$$V(\psi_1; b, \varepsilon) = \{\theta \in M(B)/|\theta(b) - \psi_1(b)| < \varepsilon\} \subset W,$$

where the equality is a definition. Let $\alpha(x)$ and f^* be related to b as in the lemma. Then $V(\psi_1; b, \varepsilon) = f^{*-1}(V(f^*(\psi_1); [x], \varepsilon))$. By Propositions 1.1 and 1.3 of [7], there are mutually disjoint M-neighborhoods $V_1^*, \ldots, V_m^*$ of the points $f^*(\psi_1), \ldots, f^*(\psi_m)$ respectively, such that $\pi_\alpha(V_i^*) = \pi_\alpha(V_1^*)$ for each i, $\pi_\alpha^{-1}(\pi_\alpha(V_1^*)) = \cup_i V_i^*$ and $V_1^* \subset V(f^*(\psi_1); [x], \varepsilon)$. The sets $U = \pi_\alpha(V_1^*)$, $V_i = f^{*-1}(V_i^*)$ satisfy (a), (b) and (c). (U is a neighborhood of $\pi(\psi_1)$ because π_α is open owing to [7, 1.2, p. 358]). In order to prove (d) fix $\phi \in U$ and $i \in \{1, \ldots, m\}$. Since $\hat{b}$ separates the points of $\pi^{-1}(\pi(\psi_1))$ and because of the definition of M-neighborhood we have

$$m(\psi_i) = M(\psi_i(b), \alpha_{\pi(\psi_1)}) = \sum_\lambda M(\lambda, \alpha_\phi) \tag{2}$$

where λ runs over the set $\{\theta(b)/\theta \in \pi^{-1}(\phi) \cap V_i\}$.

On the other hand, by part (b) of the lemma we have

$$M(\lambda, \alpha_\phi) = \sum_\eta m(\eta), \quad \text{for each } \lambda, \tag{3}$$

where η runs over the set $\{\theta \in \pi^{-1}(\phi)/\theta(b) = \lambda\}$. Now part (d) of the theorem follows from (2) and (3). Theorem 1 is thereby proved.

Using Theorem 1 and [9, 1.6, p. 677] we immediately obtain the following.

COROLLARY. $\delta(B) = \pi^{-1}(\delta(A))$.

REMARK. A point $\psi \in M(B)$ is called a singular point if π fails to be a local homeomorphism at ψ. From the (topological) conditions (a), (b) and (c) of the theorem and the uniform boundedness of the cardinal of the fibers of π, it easily follows that the (closed) set of all singular points of $M(B)$, as well as its projection, has empty interior.

3. Proof of Theorem 2. We need a lemma.

LEMMA. *If π is a homeomorphism, then $\hat{B} = \hat{A}$.*

PROOF. The same argument used at the beginning of the proof of Theorem 1 shows that we may suppose that B has a well-defined rank, say n. For $b \in B$ let $\alpha(x) = \alpha_0 + \alpha_1 x + \cdots + \alpha_{n-1}x^{n-1} + x^n$ be as in the lemma preceding Theorem 1. From (1) we obtain $\hat{b} = (-n^{-1}\alpha_{n-1})\hat{} \in A$. This is the proof.

Now we can give a proof of Theorem 2. First recall that by a strongly separable extension is meant a finitely generated projective separable extension. It is well known that $B \otimes_A B$ is a finitely generated projective extension of A. Also $M(B \otimes_A B)$ is (homeomorphic to)

$$M(B) \times_{M(A)} M(B) = \{(\psi, \theta) \in M(B) \times M(B)/\pi(\psi) = \pi(\theta)\}$$

(cf. [10, Lemma 6, p. 139]). But $M(B) \times_{M(A)} M(B)$ is a finite covering space of $M(A)$ because $M(B)$ is. Then, $C(M(B) \times_{M(A)} M(B))$ and $C(M(B))$ are strongly separable extensions of $C(M(A))$ [2, Theorem 2, p. 30]. On the other hand $C(M(B)) \otimes_{C(M(A))} C(M(B))$ is also a strongly separable extension of $C(M(A))$ [3, 1.6, p. 43]. Thus, Theorem 2 of [2] allows us to identify, via the Gelfand transform, $C(M(B)) \otimes_{C(M(A))} C(M(B))$ with $C(M(B) \times_{M(A)} M(B))$.

If e_0 is the separability idempotent for $C(M(B))$ [3, p. 40], by the Shilov idempotent theorem there is an idempotent $e \in B \otimes_A B$ such that $\hat{e} = e_0$. Set $B_0 = \{b \in B/(b \otimes 1 - 1 \otimes b)e = 0\}$. If e is written as $e = \sum_k x_k \otimes y_k$, $x_k, y_k \in B$, we show that the following relations hold:

$$\sum_{k,j} x_k \otimes x_j \otimes y_k y_j = \sum_{k,j} x_k \otimes x_j y_k \otimes y_j, \tag{4}$$

$$\sum_k x_k \otimes y_k = \sum_k y_k \otimes x_k, \tag{5}$$

$$\sum_k x_k y_k = 1. \tag{6}$$

To prove them, consider the A-algebra homomorphisms

$$\varphi_1: B \otimes_A B \to B \otimes_A B \otimes_A B, \qquad \varphi_1(x \otimes y) = x \otimes (1 \otimes y)e,$$
$$\varphi_2: B \otimes_A B \to B \otimes_A B \otimes_A B, \qquad \varphi_2(x \otimes y) = x \otimes (y \otimes 1)e.$$

Put $u_i = \varphi_i(e)$, $i = 1, 2$. By the definition of separability idempotent we have

$$\hat{u}_1 = \sum_k \hat{x}_k \otimes (1 \otimes \hat{y}_k)e_0 = \sum_k \hat{x}_k \otimes (\hat{y}_k \otimes 1)e_0 = \hat{u}_2$$

and since u_1 and u_2 are idempotents we obtain $u_1 = u_2$, which is (4).

(5) follows from the relation $\sum_k \hat{x}_k \otimes \hat{y}_k = \sum_k \hat{y}_k \otimes \hat{x}_k$ which is obtained by appealing to the uniqueness, in the commutative case, of separability idempotents.

Now consider the commutative diagram shown below, where the vertical arrows are the Gelfand transforms and the horizontal ones are defined by $x \otimes y \to xy$

$$
\begin{array}{ccc}
B \otimes_A B & \overset{\mu}{\to} & B \\
\downarrow & & \downarrow \\
C(M(B)) \otimes_{C(M(A))} C(M(B)) & \to & C(M(B))
\end{array}
$$

Since e_0 is the separability idempotent of $C(M(B))$ we have $\widehat{\mu(e)} = 1$. But $\mu(e)$ is idempotent, so $\mu(e) = 1$. This proves (6).

Next we will show that $e \in B_0 \otimes_A B_0$ by adapting an argument of Magid [11, Theorem 1.3, p. 91]. Let φ be the A-homomorphism of B into $B \otimes_A B$, $\varphi(b) = (b \otimes 1 - 1 \otimes b)e$, $b \in B$; thus the sequence

$$(0) \to B_0 \hookrightarrow B \overset{\varphi}{\to} B \otimes_A B \tag{7}$$

is exact. Tensoring with B over A we obtain the exact sequence (recall that B is A-flat)

$$(0) \to B \otimes_A B_0 \to B \otimes_A B \overset{1 \otimes \varphi}{\to} B \otimes_A B \otimes_A B.$$

By (3) $(1 \otimes \varphi)(e) = 0$; then $e \in B \otimes_A B_0$ and so we can assume that each y_k is in B_0. We set $\mu_0 = \mu / B \otimes B_0$ and $g(b) = (b \otimes 1)e$, $b \in B$, so that g is a B_0-homomorphism of B into $B \otimes_A B_0$. We have $\mu_0 \circ g = I_B$ (by (6)), which implies that B is a B_0-factor of $B \otimes_A B_0$ and hence B_0-projective. Using [1, Exercise 4, p. 147], we conclude that B_0 is a B_0-factor of B, hence an A-factor of B. Therefore B_0 is A-projective and so A-flat.

Tensoring (7) with B_0 we obtain the exact sequence

$$(0) \to B_0 \otimes_A B_0 \to B_0 \otimes_A B \overset{1 \otimes \varphi}{\to} B_0 \otimes_A B \otimes_A B.$$

By (5) $e \in B_0 \otimes_A B$ and by (4) $(1 \otimes \varphi)(e) = 0$, so $e \in B_0 \otimes_A B_0$.

We claim now that $\hat{B}_0$ separates the points of $M(B)$. Let $\psi, \theta \in M(B)$ with $\psi \neq \theta$ and $\pi(\psi) = \pi(\theta)$. Suppose that $\psi(y_k) = \theta(y_k)$ for each k. Using the fact that e_0 is the characteristic function of the set $\{(\psi', \theta') \in M(B) \times_{M(A)} M(B) / \psi' = \theta'\}$, we reach a contradiction, namely

$$0 = e_0(\psi, \theta) = \hat{e}(\psi, \theta) = \sum_k \psi(x_k)\theta(y_k) = \psi\left(\sum_k x_k y_k\right) = \psi(1) = 1.$$

Consequently $\hat{y}_k(\psi) \neq \hat{y}_k(\theta)$ for some k and the claim follows.

Therefore $M(B_0)$ is homeomorphic to $M(B)$ so the lemma tells us that $\hat{B}_0 = \hat{B}$, which gives $B = B_0 + R(B)$. This completes the proof of Theorem 2.

COROLLARY. *B is separable if and only if* $R(B) = R(A)B$ *and* π *is a local homeomorphism.*

PROOF. Assume first that B is separable. $R(B) = R(A)B$ follows from [5, Lemma 1.1, p. 78]. The fact that π is a local homeomorphism follows from [10, Theorem 5, p. 138] or by using [3, Theorem 7.1, (a) $\rightarrow$ (c), p. 72] and our Theorem 1.

For the reverse implication we consider the inertial subalgebra B_0 of B introduced in the proof of the theorem. Then $M(B_0) = M(B)$. An application of the theorem to $A' = A/R(A)$ and to $B' = B/R(B)$ (which is a finitely generated projective extension of A' because of the equality $R(B) = R(A)B$) leads to the conclusion that B' is a strongly separable extension of A'. For $\phi \in M(A)$, if $m'_\phi = \{a' \in A'/\phi(a') = 0\}$ we obtain natural C-algebra homomorphisms

$$B_0/m_\phi B_0 \xrightarrow{f} B/m_\phi B \xrightarrow{g} B'/m'_\phi B'.$$

It is easy to see that $g \circ f$ and g are isomorphisms, so f is an isomorphism. Then $\dim_C B/m_\phi B = \dim_C B_0/m_\phi B_0$. This means that B, which is a finitely generated projective extension of B_0 (see the proof of the theorem), has (a well-defined) rank 1 over B_0, that is, $B = B_0$. Then B is separable.

4. Proof of Theorem 3. Before giving the proof of the theorem it is convenient to discuss a norm on B which is easily seen to be equivalent to the norm constructed by Magid. Let (b_j, u_j), $1 \leqslant j \leqslant k$, $b_j \in B$, $u_j \in \mathrm{Hom}_A(B, A)$, be a dual basis for B over A [3, p. 4]. If we put $\|b\|' = \sum_{j=1}^k \|u_j(b)\|$, $b \in B$, we obtain a norm which induces in B the coarsest topology for which each element of $\mathrm{Hom}_A(B, A)$ is continuous. In what follows we shall suppose that each finitely generated projective extension of a Banach algebra is endowed with this Banach algebra topology.

(b) $\rightarrow$ (a). $B_1 = B \otimes_A C(\delta(A))$ is a finitely generated projective extension of $C(\delta(A))$. Using the dual basis $(b_j \otimes 1, u_j \otimes 1)$, $1 \leqslant j \leqslant k$, for B_1 over $C(\delta(A))$, we see that B is topologically isomorphic to a closed subalgebra of B_1. Also we have

$$M(B_1) \simeq \pi^{-1}(\delta(A)) = \delta(B).$$

If (b) is true, then B_1 is $C(\delta(A))$-separable (use [3, Theorem 7.1, (c) $\rightarrow$ (a), p. 72]). By [2, Theorem 2, p. 30] B_1 is isomorphic, via the Gelfand transform, to $C(\delta(B))$. But by the open mapping theorem this isomorphism is also topologic. Then B is (topologically isomorphic to) a closed separating subalgebra of $C(\delta(B))$ containing the constant functions, that is, B is a uniform algebra.

(a) $\rightarrow$ (b). If $\psi_1 \in M(B)$, $\phi = \pi(\psi_1)$ and $m(\psi_1) > 1$, there exist $b \in B$ such that $\psi_i(b) = 0$ for each $\psi_i \in \pi^{-1}(\phi)$ and $b \notin m_\phi B$. This last relation implies that not all $\phi(u_j(b))$ are 0, and so we may assume that $|\phi(u_1(b))| = 2\delta > 0$. For each $\varepsilon > 0$ there are neighborhoods V_i of ψ_i, $1 \leqslant i \leqslant \mathrm{card}\ \pi^{-1}(\phi)$, and a neighborhood U of ϕ, such that $\pi^{-1}(U) = \bigcup_i V_i$, $|\psi(b)| < \varepsilon$ for each $\psi \in \pi^{-1}(U)$, and $|\omega(u_1(b))| > \delta$ for each $\omega \in U$.

Suppose $\psi_1 \in \delta(B)$. Then $\phi \in \delta(A)$ and therefore we can find $a \in A$ with

$\|a\|_\infty = 1$ and $|a(\omega)| < \min(\varepsilon/\|b\|_\infty, 1)$, for each $\omega \notin U$. According to the hypothesis there exists an $M > 0$ such that $\|c\|_\infty \geqslant M(\Sigma_j \|u_j(c)\|_\infty)$, $c \in B$; then

$$\varepsilon \geqslant \|ab\|_\infty \geqslant M\left(\sum_j \|u_j(ab)\|_\infty\right) \geqslant M\|au_1(b)\|_\infty > M\delta$$

which is a contradiction because ε is arbitrary. Thus $\psi_1 \notin \delta(B)$ and the theorem is proved.

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