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A RUNGE THEOREM FOR SOLUTIONS OF THE HEAT EQUATION

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ABSTRACT. Let Ω_1 and Ω_2 be open sets in R^n such that $\Omega_1 \subset \Omega_2$. Every solution of the heat equation on Ω_1 admits approximation on the compact subsets of Ω_1 by functions which satisfy the heat equation throughout Ω_2 if and only if this topological condition is met: For every hyperplane π in R^n orthogonal to the time axis, every compact component of $\pi \setminus \Omega_1$ contains a compact component of $\pi \setminus \Omega_2$.

1. Introduction. If P is a linear partial differential operator on R^n and if Ω is an open set in R^n, let $P(\Omega)$ denote the kernel of the map $C^\infty(\Omega) \ni f \to Pf \in C^\infty(\Omega)$. Observe that if Ω' is an open subset of Ω then the restriction map r: $C^\infty(\Omega) \to C^\infty(\Omega')$ carries $P(\Omega)$ to a subspace of $P(\Omega')$. If $r(P(\Omega))$ is dense in $P(\Omega')$ when $P(\Omega')$ carries the topology induced by $C^\infty(\Omega')$, then the pair (Ω', Ω) is called a P-Runge pair and Ω' is called a relative P-Runge domain of Ω; cf. [5]. For a large class of elliptic operators Malgrange has proved that (Ω', Ω) is a P-Runge pair if and only every compact component of $R^n \setminus \Omega'$ contains a compact component of $R^n \setminus \Omega$ [4]. No general topological characterization of P-Runge pairs has been given for nonelliptic operators; however, B. F. Jones has proved recently that for the parabolic operator H, the heat operator on R^n, the pair (Ω, R^n) is an H-Runge pair if and only if each hyperplane in R^n orthogonal to the time axis intersects $R^n \setminus \Omega$ to form a set that is free of compact components [3]. We shall generalize the theorem of Jones by characterizing all H-Runge pairs.

2. Statement of results. Let H denote the heat operator on R^n, $n \geq 2$, and let Ω_1 and Ω_2 be open sets in R^n such that $\Omega_1 \subset \Omega_2$. Our main result is

THEOREM 1. *If, for each hyperplane π in R^n orthogonal to the time axis, every compact component of $\pi \setminus \Omega_1$ contains a compact component of $\pi \setminus \Omega_2$, then (Ω_1, Ω_2) is an H-Runge pair.*

Modifying the work of Jones in a straightforward fashion, we also prove the converse.

THEOREM 2. *If (Ω_1, Ω_2) is an H-Runge pair then, for each hyperplane π orthogonal to the time axis, every compact component of $\pi \setminus \Omega_1$ contains a compact component of $\pi \setminus \Omega_2$.*

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3. Proof of the main theorem. The Hahn-Banach theorem implies that (Ω_1, Ω_2) is an H-Runge pair if and only if each distribution $\mu \in \mathcal{E}'(\Omega_1)$ that annihilates $H(\Omega_2)$ also annihilates $H(\Omega_1)$. If μ does annihilate $H(\Omega_2)$, μ annihilates in particular the exponential-polynomial members of $H(R^n)$, and a theorem of Malgrange implies that $\mu = {}^{\dagger}H\nu$ for some $\nu \in \mathcal{E}'(R^n)$ where ${}^{\dagger}H$ denotes the operator formally adjoint to H [4]. If ν is supported in Ω_1 there exists a cut-off function φ such that for each $f \in H(\Omega_1)$ the function φf agrees with f on a neighborhood of supp ν and $\varphi f \in \mathcal{E}(R^n)$; consequently, supp $\nu \cap \operatorname{supp} H(\varphi f) = \varnothing$ and

$$\langle \mu, f \rangle = \langle {}^{\dagger}H\nu, f \rangle = \langle {}^{\dagger}H\nu, \varphi f \rangle = \langle \nu, H(\varphi f) \rangle = 0. \tag{3.1}$$

Thus, μ annihilates $H(\Omega_1)$ if ν is supported in Ω_1. The remainder of this section deals with the complications which arise when ν is not supported in Ω_1. We shall prove that supp $\nu \setminus \Omega_1$ is a compact subset of $\partial\Omega_1$ on which ν vanishes to infinite order; then, approximating ν by functions which are supported in Ω_1, we shall apply (3.1) to conclude that μ does indeed annihilate $H(\Omega_1)$.

Let E be a fundamental solution for the operator ${}^{\dagger}H$. For each multi-index $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$,

$$D^{\alpha}\nu = D^{\alpha}\nu * \delta = D^{\alpha}\nu * {}^{\dagger}HE = {}^{\dagger}HD^{\alpha}\nu * E = {}^{\dagger}H\nu * D^{\alpha}E = \mu * D^{\alpha}E.$$

The hypoellipticity of ${}^{\dagger}H$ implies that for each $x \in R^n \setminus \operatorname{supp} \mu$

$$D^{\alpha}\nu(x) = \langle \mu, D^{\alpha}E(x - \cdot) \rangle \tag{3.2}$$

where $D^{\alpha}E(x - \cdot)$ denotes the function $y \to D^{\alpha}E(x - y)$. Since

$$HD^{\alpha}E(x - \cdot) = {}^{\dagger}HD^{\alpha}E(x - \cdot) = D^{\alpha}\,{}^{\dagger}HE(x - \cdot) = D^{\alpha}\delta(x - \cdot),$$

$D^{\alpha}E(x - \cdot) \in H(\Omega_2)$ for each $x \notin \Omega_2$. The hypothesis that μ annihilates $H(\Omega_2)$ and (3.2) imply that ν vanishes to infinite order on $R^n \setminus \Omega_2$.

Let π be any hyperplane in R^n orthogonal to the time axis. Since $D^{\alpha}\nu$ satisfies ${}^{\dagger}H(D^{\alpha}\nu) = 0$ on $R^n \setminus \operatorname{supp} \mu$, $D^{\alpha}\nu$ is real-analytic in the space variables on $R^n \setminus \operatorname{supp} \mu$. Thus, $D^{\alpha}\nu$ vanishes on every component of $\pi \setminus \operatorname{supp} \mu$ that contains an infinite order zero of ν. Every unbounded component of $\pi \setminus \operatorname{supp} \mu$ contains such a zero because ν has compact support. Every bounded component of $\pi \setminus \operatorname{supp} \mu$ that contains a compact component of $\pi \setminus \Omega_1$ must contain a compact component of $\pi \setminus \Omega_2$ by the hypothesis of Theorem 1; consequently, $D^{\alpha}\nu$ vanishes on every component of $\pi \setminus \operatorname{supp} \mu$ that contains a component of $\pi \setminus \Omega_1$. Since π and α are arbitrary, ν must vanish to infinite order throughout $R^n \setminus \Omega_1$. Clearly then supp $\nu \subset \bar{\Omega}_1$, and ν vanishes to infinite order on supp $\nu \setminus \Omega_1$. The closed set $K = \operatorname{supp} \nu \setminus \Omega_1$ is compact because ν has compact support.

Henceforth we shall write R^n as $R^{n-1} \times R$ so that we may distinguish between the one time variable and the $n - 1$ space variables on which solutions of the heat equation depend. Let $\{D_i\}_{i=1}^{m}$ be a finite collection of mutually disjoint, connected open sets such that

(i) each D_i is a product of a subset of R^{n-1} and a subinterval of R,

(ii) $\bar{D}_i \cap \bar{D}_j$ is contained in a hyperplane orthogonal to the time axis and has finitely many components for $i \neq j$; and

(iii) $F = \bigcup_{i=1}^{m} \bar{D}_i$ is a compact neighborhood of K disjoint from supp μ.

Let φ be a cut-off function supported in the interior of F such that $\varphi \equiv 0$ on a neighborhood of each component of $\bar{D}_i \cap \bar{D}_j$ disjoint from K and such that $\varphi \equiv 1$ on a neighborhood of K. Finally, let $Z = \{x \notin \text{supp}\, \mu | \nu$ vanishes to infinite order at $x\}$. Any $\bar{D}_i \cap \bar{D}_j$ has its components contained in a copy of R^{n-1} when $i \neq j$; and since ν is real-analytic in the space variables on a neighborhood of $\bar{D}_i \cap \bar{D}_j$, ν vanishes to infinite order throughout any component of $\bar{D}_i \cap \bar{D}_j$ meeting K. Thus $\varphi\nu$ vanishes to infinite order throughout $\bar{D}_i \cap \bar{D}_j$ when $i \neq j$. Since each boundary point of D_i is either contained in some $\bar{D}_j$ or is a boundary point of F, $\varphi\nu$ vanishes to infinite order throughout the boundary of D_i. Clearly then $\varphi\nu$ vanishes to infinite order throughout the boundary of each component of $D_i \setminus Z$.

Now, for each $f \in H(\Omega_1)$,

$$\begin{aligned}\langle \mu, f\rangle &= \langle (1-\varphi)\mu + \varphi\mu, f\rangle,\\ &= \langle {}^{\dagger}H[(1-\varphi)\nu], f\rangle + \langle {}^{\dagger}H(\varphi\nu), f\rangle,\\ &= \langle {}^{\dagger}H(\varphi\nu), f\rangle,\end{aligned}$$

since $Hf \equiv 0$ on a neighborhood of supp$(1-\varphi)\nu$. Recall that $\varphi\nu$ is smooth and supported in F. Evidently, then,

$$\langle \mu, f\rangle = \int_F f\, {}^{\dagger}H(\varphi\nu) = \sum_{i=1}^{m} \int_{D_i} f\, {}^{\dagger}H(\varphi\nu) = \sum_{i=1}^{m} \int_{D_i\setminus Z} f\, {}^{\dagger}H(\varphi\nu).$$

Let $A \times (a, b)$ be an arbitrary component of some $D_i \setminus Z$, where $A \subset R^{n-1}$ and $(a, b) \subset R$. We shall prove that $\int_{A\times(a,b)} f\, {}^{\dagger}H(\varphi\nu) = 0$, from which it follows that μ annihilates $H(\Omega_1)$.

Note that since $\varphi\nu$ vanishes to infinite order throughout the boundary of $A \times (a, b)$ the function $\overline{\varphi\nu}$ which agrees with $\varphi\nu$ on $A \times (a, b)$ and which is zero outside $A \times (a, b)$ is smooth throughout $R^{n-1} \times R$. Composing $\overline{\varphi\nu}$ with a translation of $R^{n-1} \times R$, we may assume that $(0, 0) \in A \times (a, b)$. Now let $B \times (c, d)$ be a connected open neighborhood of $\bar{A} \times [a, b]$ such that $\bar{B} \times [c, d]$ is a subset of $(R^{n-1} \times R) \setminus \text{supp}\, \mu$. Clearly $B \times (a, b)$ is disjoint from Z because $A \times (a, b)$ is disjoint from Z. For each $t > 1$ the dilation $T_t\colon R^{n-1} \times R \to R^{n-1} \times R$ defined by $(\xi, \tau) \to T_t(\xi, \tau) = (t\xi, t^2\tau)$ is such that $R^{n-1} \times [a, b] \subset T_t(R^{n-1} \times (a, b))$; thus, $T_t^*(\overline{\varphi\nu}) = \overline{\varphi\nu} \circ T_t$ is supported in $R^{n-1} \times (a, b)$. For all t sufficiently close to 1, $T_t(\text{supp}\, \overline{\varphi\nu})$ is contained in $B \times (c, d)$. For such t we have supp $T_t^*(\overline{\varphi\nu}) = T_t^{-1}(\text{supp}\, \overline{\varphi\nu}), \subset B \times (c, d) \cap R^{n-1} \times (a, b), \subset B \times (a, b)$. Thus supp $T_t^*(\overline{\varphi\nu})$ is disjoint from Z. In particular, supp $T_t^*(\overline{\varphi\nu})$ is disjoint from $R^n \setminus \Omega_1$. Since ${}^{\dagger}HT_t^*(\overline{\varphi\nu}) = t^2T_t^*({}^{\dagger}H\overline{\varphi\nu})$,

$$\lim_{t\to 1^+} \langle f, {}^{\dagger}HT_t^*(\overline{\varphi\nu})\rangle = \langle f, {}^{\dagger}H\,\overline{\varphi\nu}\rangle = \int_{A\times(a,b)} f\, {}^{\dagger}H(\varphi\nu).$$

On the other hand, since ${}^{\dagger}HT_t^*(\overline{\varphi\nu})$ is supported in Ω_1,

$$\langle f, {}^{\dagger}HT_t^*(\overline{\varphi\nu})\rangle = \langle Hf, T_t^*(\overline{\varphi\nu})\rangle = \langle 0, T_t^*(\overline{\varphi\nu})\rangle = 0.$$

Thus, $\int_{A\times(a,b)} f\, {}^{\dagger}H(\overline{\varphi\nu}) = 0$ and μ annihilates $H(\Omega_1)$. Q.E.D.

4. The converse to the main theorem. Our proof that the topological condition on Ω_1 and Ω_2 is necessary for (Ω_1, Ω_2) to be an H-Runge pair is a modification of the proof given in [3] for the special case $\Omega_2 = R^n$. This proof requires the Tychonoff counterexample to uniqueness to the initial-value problem for the heat equation, which is a nonzero element of $H(R^n)$ supported in a strip bounded by two hyperplanes orthogonal to the time axis. The existence of such a function is proved in [2].

We shall prove that if for some hyperplane π orthogonal to the time axis some compact component of $\pi \setminus \Omega_1$ is contained in Ω_2, then there exists a $\mu \in \mathcal{E}'(\Omega_1)$ such that μ annihilates $H(\Omega_2)$ but such that μ does not annihilate $H(\Omega_1)$; the Hahn-Banach theorem then implies that $H(\Omega_2)$ is not dense in $H(\Omega_1)$. We may assume that $\pi = R^{n-1} \times \{0\}$ and identify π with R^{n-1}. Now, there exists an open set G in R^{n-1} contained in Ω_2 and a nonempty compact subset K of $\pi \setminus \Omega_1$ such that $G \setminus \Omega_1 = K$ [1]. Choose $\varphi \in \mathcal{D}(G)$ such that $\varphi \equiv 1$ on a neighborhood of K in R^{n-1}. There exists a $\delta > 0$ such that $\operatorname{supp} \varphi \times [-\delta, \delta] \subset \Omega_2$ and such that $\operatorname{supp}\|\nabla \varphi\| \times [-\delta, \delta] \subset \Omega_1$, where ∇ is the $(n-1)$-dimensional gradient. By suitably rescaling the Tychonoff counterexample to uniqueness and reversing the sign of the time variable, we can construct a function g supported in $R^{n-1} \times [-\delta, \delta]$ such that ${}^{\dagger}Hg = 0$ throughout R^n and g is nonzero at some point x_0 in K. The function ${}^{\dagger}H(\varphi g)$ is supported in $\operatorname{supp}\|\nabla\varphi\| \times [-\delta, \delta] \subset \Omega_1$, and φg is supported in $\operatorname{supp} \varphi \times [-\delta, \delta] \subset \Omega_2$. For each $f \in H(\Omega_2)$ one has

$$\langle {}^{\dagger}H(\varphi g), f\rangle = \langle g, \varphi H f\rangle = \langle g, 0\rangle = 0.$$

However, if E is any fundamental solution for the heat equation then $E(\cdot - x_0) \in H(\Omega_1)$, yet

$$\langle {}^{\dagger}H(\varphi g), E(\cdot - x_0)\rangle = \langle \varphi g, HE(\cdot - x_0)\rangle = \langle \varphi g, \delta(\cdot - x_0)\rangle = g(x_0) \neq 0.$$

Thus $\mu = {}^{\dagger}H(\varphi g)$ annihilates $H(\Omega_2)$ but does not annihilate $H(\Omega_1)$; and $H(\Omega_2)$ is not dense in $H(\Omega_1)$. Q.E.D.

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