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S-SETS AND S-PERFECT MAPPINGS

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ABSTRACT. In this note we generalize the notion of an S-closed space to S-sets of a space. Among our characterizations of S-sets, we show that a subset A of an extremally disconnected space X is an S-set in X if and only if A is an H-set in X. We also investigate conditions under which mappings or their inverses preserve S-sets.

1. Introduction. For convenience, all spaces are assumed to be Hausdorff although many of the assertions made below apply in a more general setting. Concerning notation, $\bar{A}$ (A°) will be used to denote the closure (interior) of A in a space X, and when necessary, $\bar{A}^\tau$ will denote the closure of A in the space X endowed with the topology τ. If $x \in X$, $\mathfrak{n}_x$ is the open neighborhood filter at x and $\bar{\mathfrak{n}}_x \equiv \{\bar{N} | N \in \mathfrak{n}_x\}$. Correspondingly, $\mathbb{S}_x$ (and, when necessary, $\mathbb{S}_x(X, \tau)$) will denote the filter of all semi-open (abbreviated s.o.) subsets of X which contain x. A subset S of X is *semi-open* if there exists an open subset O of X such that $O \subseteq S \subseteq \bar{O}$. $\bar{\mathbb{S}}_x$ is defined analogously and the family of all s.o. subsets of a space X is denoted $\mathbb{S}(X)$.

The concept of an H-set was introduced by N. Veličko in [9]. A subset A of a Hausdorff space X is an *H-set* if every cover of A by open subsets of X contains a finite dense subsystem, i.e. a finite subfamily whose closures in X cover A. This concept was independently introduced in [6] and called *H-closed relative to X*. A Hausdorff space X is *H-closed* if it is an H-set (relative to X). Veličko showed that θ-closed subsets of H-closed spaces are H-sets (a subset A of X is *θ-closed* if $A = \mathrm{cl}_\theta A \equiv \{x \in X | \bar{N} \cap A \neq \varnothing$ for every $N \in \mathfrak{n}_x\})$ and that H-sets of H-closed Urysohn spaces are θ-closed.

More recently, T. Thompson defined a space X to be *S-closed* if every s.o. cover of X contains a finite dense subsystem. A subset A of a Hausdorff space X will be called an *S-set* (relative to X) if every cover of A by s.o. subsets of X contains a finite dense subsystem.

In §2, we present some characterizations of S-sets and obtain results analogous to those of Veličko mentioned above. We also define an S-point. This concept was introduced by E. K. van Douwen in [8] who noted that remote points of a completely regular space X are S-points of βX. We will show that extremally

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disconnected spaces consist entirely of S-points. (A space X is *extremally disconnected*, abbreviated e.d., if regular closed subsets of X are open; a subset A of X is *regular closed* if $A = \bar{O}$ for some open subset O of X.)

In §3, we introduce the definition of s-perfect mapping and we consider the preservation of S-sets by mappings and their inverses. We produce a large class of s-perfect mappings and exhibit a class of spaces which possess precisely the same S-sets. Through the use of S-points we obtain necessary conditions for a mapping to be s-perfect, and, as a result, obtain conditions under which the domain of an s-perfect mapping is e.d.

In §4, we present a characterization of compact e.d. spaces which enables us to observe that the class of s-perfect mappings contains a well-known subclass. We also note conditions under which the image of a compact e.d. space is necessarily e.d.

2. S-sets. For a subset A of X, the s-closure of A, denoted $\mathrm{cl}_s A$, is the set $\{x \in X \mid \bar{S} \cap A \neq \varnothing$ for every $S \in \mathbb{S}_x\}$. If A is a subset of X and $\mathcal{F}$ is a filter base on X we say that $\mathcal{F}$ *meets* A if $F \cap A \neq \varnothing$ for every $F \in \mathcal{F}$. $\mathcal{F}$ is said to *s-accumulate at x* (x is an s-accumulation point of $\mathcal{F}$), written $x \in \operatorname{sad} \mathcal{F}$, if $\mathcal{F}$ meets each $\bar{S} \in \bar{\mathbb{S}}_x$, and $\mathcal{F}$ *s-converges to x* if every $\bar{S} \in \bar{\mathbb{S}}_x$ contains some $F \in \mathcal{F}$. Correspondingly, we say that $\mathcal{F}$ *s_*-accumulates at x*, written $x \in \operatorname{sad}_* \mathcal{F}$, if $\mathcal{F}$ meets each $S \in \mathbb{S}_x$.

A straightforward application of Zorn's Lemma yields

LEMMA 2.1. *Let A be a nonempty subset of a space X. If $\mathcal{F}$ is a filter base on X which meets A, then $\mathcal{F}$ is contained in a maximal filter base which also meets A.*

PROPOSITION 2.2. *The following are equivalent for a space X.*

(i) *A is an S-set.*

(ii) *Every maximal filter base on X which meets A s-converges to some point in A.*

(iii) *Every filter base on X which meets A s-accumulates at some point in A.*

(iv) *Every open filter base on X which meets A s_*-accumulates at some point in A.*

(v) *If $\mathcal{F} = \{F_\alpha\}_{\alpha \in \Lambda}$ is an open filter base on X which meets A, then $(\bigcap_\alpha (\bar{F}_\alpha)^\circ) \cap A \neq \varnothing$.*

PROOF. (i) $\Rightarrow$ (ii). Let A be an S-set and suppose $\mathcal{U}$ is a maximal filter base on X which meets A and does not s-converge to some point in A. Then, if $x \in A$, there exists $S_x \in \mathbb{S}_x$ such that $U \cap (X \setminus \bar{S}_x) \neq \varnothing$ for every $U \in \mathcal{U}$. The maximality of $\mathcal{U}$ implies that $\mathcal{U} \supseteq \{U \cap A \mid U \in \mathcal{U}\}$ and hence also that $\mathcal{U} \supseteq \{U \cap (X \setminus \bar{S}_x) \mid U \in \mathcal{U}\}$. Thus there exists $U_x \in \mathcal{U}$ such that $U_x \cap \bar{S}_x = \varnothing$. $\{\bar{S}_x\}_{x \in A}$ is an s.o. cover of A, and A is therefore contained in some $\bigcup_{i=1}^n \bar{S}_{x_i}$. Now $\bigcap_{i=1}^n U_{x_i} \in \mathcal{U}$ and $(\bigcap_{i=1}^n U_{x_i}) \cap A \subseteq (\bigcap_{i=1}^n U_{x_i}) \cap (\bigcup_{i=1}^n \bar{S}_{x_i}) = \varnothing$, a contradiction since $\mathcal{U}$ meets A. This establishes (ii). Other implications are straightforward and therefore omitted.

PROPOSITION 2.3. *In an e.d. space every H-set is an S-set.*

PROOF. If $\{S_\alpha\}_{\alpha \in \Lambda}$ is an s.o. cover of an H-set, A, then $\{\bar{S}_\alpha\}_{\alpha \in \Lambda}$ is an open cover of A and some finite union must contain A. Hence A is an S-set.

REMARK 2.4. Since S-closed spaces are e.d. [7, Theorem 7], an H-closed space is e.d. if and only if every H-set is an S-set. However, $\beta N \setminus N$ is an H-set relative to the e.d. space βN which is not e.d. [2, 6R(1)]. Thus S-sets need not be e.d.

COROLLARY 2.5. *A subset A of a S-closed space X is an S-set if and only if* $A = \mathrm{cl}_\theta A$.

PROOF. Note that X is an e.d. Urysohn space so that A is an H-set if and only if $A = \mathrm{cl}_\theta A$ [9]. Thus by Proposition 2.3 A is an S-set if and only if $A = \mathrm{cl}_\theta A$.

A point $x \in X$ will be called an *S-point* (of X) if $x \in (\bar{S})^\circ$ for every $S \in \mathbb{S}_x$. $x \in X$ is called an *ordinary point* if x is not an S-point.

REMARK 2.6. It is easily seen that x is an S-point of X if and only if X is e.d. at x [8]. For if $x \notin (\bar{S})^\circ$ where $S \in \mathbb{S}_x$, then $x \in (\overline{\bar{S}^\circ}) \cap (\overline{X \setminus \bar{S}})$ so that X is not e.d. at x, and if $x \in \bar{U} \cap \bar{V}$ where U, V are disjoint and open in X, then $x \notin (\bar{U})^\circ$ so that x is not an S-point of X. Thus a space X is e.d. if and only if every $x \in X$ is an S-point of X. Note that for an arbitrary subset A of X, $\mathrm{cl}_\theta A \setminus \mathrm{cl}_s A$ does not contain any S-points. Thus $\mathrm{cl}_\theta A = \mathrm{cl}_s A$ whenever $\mathrm{cl}_\theta A \setminus A$ consists entirely of S-points as, for example, in e.d. spaces.

3. S-perfect mappings. A mapping f: $X \to Y$ is said to be *irresolute* [1] if given $S \in \mathbb{S}(Y)$, $f^{-1}(S) \in \mathbb{S}(X)$.

PROPOSITION 3.1. *If f: $X \to Y$ is a continuous irresolute mapping, then f preserves S-sets.*

PROOF. Let A be an S-set in X and let $\{S_\alpha\}_{\alpha \in \Lambda}$ be an s.o. cover of $f(A)$. Then $\{f^{-1}(S_\alpha)\}_{\alpha \in \Lambda}$ is an s.o. cover of A so that A is contained in some $\bigcup_{i=1}^n \overline{f^{-1}(S_{\alpha_i})}$. Hence

$$f(A) \subseteq \bigcup_{i=1}^{n} f\left(\overline{f^{-1}(S_{\alpha_i})}\right) \subseteq \bigcup_{i=1}^{n} \bar{S}_{\alpha_i}$$

which proves our assertion.

A mapping f: $X \to Y$ is called an *s-closed mapping* if $\mathrm{cl}_s f(A) \subseteq f(\mathrm{cl}_s A)$ for every subset A of X, and f is said to be *s-perfect* if f is an s-closed mapping and point inverses are S-sets.

LEMMA 3.2. *A mapping f: $X \to Y$ is s-perfect if and only if* $\mathrm{sad}\, f(\mathfrak{F}) \subseteq f(\mathrm{sad}\, \mathfrak{F})$ *for every filter base $\mathfrak{F}$ on X.*

PROOF OF NECESSITY. Let $\mathfrak{F}$ be a filter base on X and let $y \in Y \setminus f(\mathrm{sad}\, \mathfrak{F})$. For $x \in f^{-1}(y)$, there exist $S_x \in \mathbb{S}_x$ and $F_x \in \mathfrak{F}$ such that $\bar{S}_x \cap F_x = \varnothing$. $\{S_x\}_{x \in f^{-1}(y)}$ is an s.o. cover of the S-set, $f^{-1}(y)$, so that $f^{-1}(y)$ is contained in some $\bigcup_{i=1}^n \bar{S}_{x_i}$. $F = \bigcap_{i=1}^n F_{x_i} \in \mathfrak{F}$ and $F \cap (\bigcup_{i=1}^n \bar{S}_{x_i}) = \varnothing$. Hence $\mathrm{cl}_s F \cap f^{-1}(y) = \varnothing$. Since f is an s-closed mapping, $y \notin \mathrm{cl}_s f(F)$ and therefore, $y \notin \mathrm{sad}\, f(\mathfrak{F})$.

PROOF OF SUFFICIENCY. Let A be a subset of X and let $y \in \mathrm{cl}_s f(A)$. $\mathfrak{F} = \{F \subseteq X | A \subseteq F\}$ is a filter base on X such that $y \in \mathrm{sad}\, f(\mathfrak{F}) \subseteq f(\mathrm{sad}\, \mathfrak{F})$. Hence $\varnothing \neq \mathrm{sad}\, \mathfrak{F} \cap f^{-1}(y) \subseteq \mathrm{cl}_s A \cap f^{-1}(y)$. This proves that f is an s-closed mapping.

We shall use Proposition 2.2 to show that point inverses are S-sets. Let $\mathfrak{F}$ be a filter base on X which meets $f^{-1}(y)$. Then $y \in f(F)$ for every $F \in \mathfrak{F}$ so that $y \in \text{sad}\, f(\mathfrak{F}) \subseteq f(\text{sad}\, \mathfrak{F})$. Hence sad $\mathfrak{F} \cap f^{-1}(y) \neq \varnothing$. By Proposition 2.2, $f^{-1}(y)$ is an S-set. This completes the proof.

PROPOSITION 3.3. *If f: $X \to Y$ is s-perfect, then inverse images of S-sets are S-sets.*

PROOF. We shall use Proposition 2.2. Let A be an S-set in Y and let $\mathfrak{F}$ be a filter base on X which meets $f^{-1}(A)$. Set $\mathcal{G} = \{F \cap f^{-1}(A) | F \in \mathfrak{F}\}$. Then $f(\mathcal{G})$ is a filter base on Y which meets A and $f(\text{sad}\, \mathcal{G}) \cap A \supseteq \text{sad}\, f(\mathcal{G}) \cap A \neq \varnothing$ (Proposition 2.2 and Lemma 3.2). Thus sad $\mathfrak{F} \cap f^{-1}(A) \supseteq$ sad $\mathcal{G} \cap f^{-1}(A) \neq \varnothing$ so that $f^{-1}(A)$ is an S-set by Proposition 2.2.

PROPOSITION 3.4. *If f: $X \to Y$ is a continuous surjection and X is compact and e.d., then f is s-perfect.*

PROOF. Let $\mathfrak{F}$ be a filter base on X and let $y \in \text{sad}\, f(\mathfrak{F})$. Then $\mathcal{G} = \{f^{-1}(\overline{N}) \cap F | N \in \mathfrak{n}_y, F \in \mathfrak{F}\}$ is a filter base on X, and since X is S-closed, $\mathcal{G}$ s-accumulates at some $x \in X$ [7, Theorem 2]. $x \in \text{sad}\, \mathcal{G} \subseteq \text{sad}\, \mathfrak{F}$. We show that $x \in f^{-1}(y)$. Suppose $x \notin f^{-1}(y)$. Since f is a surjection, Y is compact and there exist $N_y \in \mathfrak{n}_y$, $N_{f(x)} \in \mathfrak{n}_{f(x)}$ such that $\overline{N}_y \cap \overline{N}_{f(x)} = \varnothing$. The continuity of f implies there exists $N_x \in \mathfrak{n}_x$ such that $f(\overline{N}_x) \subseteq \overline{N}_{f(x)}$. But $x \in \text{sad}\, \mathcal{G}$ so that $\overline{N}_x \cap f^{-1}(\overline{N}_y) \neq \varnothing$. Therefore $f(\overline{N}_x) \cap \overline{N}_y \neq \varnothing$, a contradiction. We conclude that $x \in f^{-1}(y)$ so that $y \in f(\text{sad}\, \mathfrak{F})$ and f is s-perfect (Lemma 3.2).

COROLLARY 3.5. *If f: $X \to Y$ is a continuous mapping of an S-closed space into a Urysohn space, then f is s-perfect.*

PROOF. The proof is similar to the proof of Proposition 3.4.

Let X be a fixed set and let $\mathfrak{T}$ be the family of all topologies on X having the same family of regular open subsets of X (a subset R of X is *regular open* if $R = C^0$ for some regular closed subset C of X). The elements of $\mathfrak{T}$ are said to be *r.o.-equivalent*.

PROPOSITION 3.6. *If τ, σ are r.o.-equivalent topologies on a space X, the identity mapping i: $(X, \tau) \to (X, \sigma)$ is s-perfect.*

PROOF. Point inverses are singletons and are therefore S-sets. We show that i is an s-closed mapping.

Let A be a subset of X and suppose $i(x) \notin i(\text{cl}_s A)$. Then $x \notin \text{cl}_s A$ and there exists $S \in \mathbb{S}_x(X, \tau)$ such that $\bar{S}^\tau \cap A = \varnothing$. By 1.1 of [5], $\bar{S}^\tau = \bar{S}^\sigma \in \mathbb{S}_{i(x)}$ and $\varnothing = \bar{S}^\tau \cap A = \bar{S}^\sigma \cap A$. Thus $i(x) \notin \text{cl}_s(i(A))$ and the proof is complete.

COROLLARY 3.7. *If τ, σ are r.o.-equivalent topologies on a space X, then a subset A of X is an S-set relative to (X, τ) if and only if A is an S-set relative to (X, σ).*

PROOF. Let i: $(X, \tau) \to (X, \sigma)$ be the identity mapping so that i^{-1}: $(X, \sigma) \to (X, \tau)$ is also the identity mapping. By Proposition 3.6, i and i^{-1} are s-perfect mappings and by Proposition 3.3, both preserve S-sets.

REMARK 3.8. Let $g: X \to Xs$ denote the semiregularization of a space X. Then X and Xs are r.o.-equivalent, g and g^{-1} are s-perfect, and A is an S-set in X if and only if A is an S-set in Xs.

N. Levine has shown that $S(X, \tau) = S(X, \sigma)$ implies $\tau = \sigma$ [4]. We conclude that a bijective semi-open and irresolute mapping is a homeomorphism. Compare [1].

PROPOSITION 3.9. *Suppose $f: X \to Y$ maps an ordinary point to an S-point. If either* (i) *f is continuous or* (ii) *Y is S-closed, then f is not s-perfect.*

PROOF. Let x be an ordinary point of X whose image, $y = f(x)$, is an S-point. There exists $S \in \mathfrak{S}_x$ such that $x \notin (\bar{S})^\circ$. $\mathfrak{F} = \{N \cap S^\circ | N \in \mathfrak{n}_x\}$ is a filter base on X. sad $\mathfrak{F} \subseteq$ sad $\mathfrak{n}_x \subseteq \{x\}$; but $S_x = X \setminus (\bar{S})^\circ \in \mathfrak{S}_x$ and for $N \in \mathfrak{n}_x$, $\varnothing = \bar{S}_x \cap (N \cap (\bar{S})^\circ) \supseteq \bar{S}_x \cap (N \cap S^\circ)$. Thus sad $\mathfrak{F} = \varnothing$.

If Y is S-closed, $f(\mathfrak{F})$ s-accumulates at some $z \in Y$; and, if f is continuous, $f(\mathfrak{F})$ s-converges to y (y is an S-point). In either case, sad $f(\mathfrak{F}) \neq \varnothing = f(\text{sad } \mathfrak{F})$. By Lemma 3.2, f is not s-perfect.

COROLLARY 3.10. *If $f: X \to Y$ is an s-perfect mapping and either* (i) *f is continuous and Y is e.d. or* (ii) *Y is S-closed, then X is e.d.*

PROOF. Y is e.d., so that Remark 2.6 and Proposition 3.9 imply that every $x \in X$ is an S-point. Therefore X is e.d. by Remark 2.6.

COROLLARY 3.11. *If $f: X \to Y$ is a continuous mapping of an H-closed space into an S-closed space, then f is s-perfect if and only if X is e.d.*

PROOF. Sufficiency is given by Corollary 3.5 and necessity follows from Corollary 3.10.

4. Related results. In [7], Thompson showed that every compact e.d. space is S-closed. We have shown that every H-closed e.d. space is S-closed and it is well known that there exist noncompact, H-closed e.d. spaces [5]. Thus the class of S-closed spaces properly contains the class of compact e.d. spaces.

G. Viglino defined a space to be *C-compact* if every closed subset of X is an H-set. In so doing, Viglino introduced a class of spaces which properly contains the class of compact spaces and is properly contained in the class of minimal Hausdorff (and hence H-closed) spaces [10]. It therefore seems natural to ask whether or not a new class of noncompact spaces, which is properly contained in the class of S-closed spaces, can be found in similar fashion. Our next proposition provides a negative answer to this question.

PROPOSITION 4.1. *Every closed subset of a space X is an S-set if and only if X is compact and e.d.*

PROOF OF SUFFICIENCY. If C is a closed subset of a compact e.d. space and $\{S_\alpha\}_{\alpha \in \Lambda}$ is an s.o. cover of C, then $\{\bar{S}_\alpha\}_{\alpha \in \Lambda}$ is an open cover of a compact set so that C is contained in some finite union of elements.

PROOF OF NECESSITY. If every closed subset of X is an S-set, then, in particular, X is S-closed so that X is e.d. Thus X is a Urysohn space. Moreover, every closed subset of X is an H-set so that X is normal [**6**, Theorem 3.3]. Thus X is compact by [**6**, (2.17)] and the proof is complete.

COROLLARY 4.2. *The continuous irresolute image of a compact e.d. space is compact and e.d.*

PROOF. The compactness of a continuous image of a compact space is well known. To obtain the e.d. property, apply Propositions 3.1 and 4.1.

REMARK 4.3. A. Gleason proved that every compact space is the continuous perfect irreducible image of a compact e.d. space [**3**]. Proposition 3.4 and Corollary 4.2 show that the associated mappings are necessarily s-perfect but need not be irresolute.

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