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GROUP RINGS WHOSE TORSION UNITS FORM A SUBGROUP

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ABSTRACT. Denote by $TU(\mathbf{Z}G)$ the set of units of finite order of the integral group ring of a group G. We determine the class of all groups G such that $TU(\mathbf{Z}G)$ is a subgroup and study how this property relates to certain properties of the unit groups.

1. Introduction. Let G be a group. We denote by $\mathbf{Z}G$ its integral group ring and by $U(\mathbf{Z}G)$ the group of units of this ring. Also, we shall denote by $T = T(G)$ and $TU(\mathbf{Z}G)$ the set of all elements of finite order in G and $U(\mathbf{Z}G)$ respectively.

S. K. Sehgal and H. J. Zassenhaus have determined the classes of all groups G such that $U(\mathbf{Z}G)$ is a nilpotent or an FC group in [3] and [4]. If restricted to finite groups, it is easy to see that both classes coincide.

M. M. Parmenter and C. Polcino Milies have shown in [1] that, in the finite case, the characterization of such groups G follows from the fact that, in both cases, $TU(\mathbf{Z}G)$ forms a subgroup.

In this note, we determine all groups G which are either nilpotent or FC and such that $TU(\mathbf{Z}G)$ is a subgroup. It will follow that this property is not enough to lead either to nilpotency or the FC property when G is infinite.

2. The Theorem. We begin with

LEMMA. *Let G be a group such that $TU(\mathbf{Z}G)$ forms a subgroup. Then $TU(\mathbf{Z}G) = \pm T$, i.e. every unit of finite order is trivial.*

PROOF. Let $u \in TU(\mathbf{Z}G)$ with $o(u) = p_1^{n_1} \dots p_t^{n_t}$. We shall show that $u \in \pm T$ by induction on t.

If $t = 1$ then u is a p-element for some rational prime p; hence, there is an element $g \in \operatorname{supp}(u)$ such that $o(g)$ is finite [2, Theorem VI.2.1]. Since $TU(\mathbf{Z}G)$ is a subgroup, $v = g^{-1} \cdot u$ is a unit of finite order and, writing $v = \sum_{a \in G} v(a)a$, we have that $v(1) \neq 0$. It follows from [2, Corollary II.1.2] that $v = \pm 1$, thus $u = \pm g \in T$.

Now, we assume that the result holds if $o(u)$ is divisible at most by $t - 1$ different primes and let $o(u) = p_1^{n_1} \dots p_t^{n_t}$.

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Writing $o(u) = mp_t^{n_t}$ where $\gcd(m, p_t^{n_t}) = 1$, there exist integers r, s such that $m + sp_t^{n_t} = 1$. Set $u_1 = u^{rm}$ and $u_2 = u^{sp_t^{n_t}}$. Then $u = u_1 \cdot u_2$ and $u_1, u_2 \in \pm T$ by the induction hypothesis. $\square$

We can now prove our main result.

THEOREM. *Let G be a group such that $TU(\mathbf{Z}G)$ is a subgroup of $U(\mathbf{Z}G)$. Then T is a subgroup of G and one of the following conditions holds.*

(i) *T is abelian and, for all $x \in G$ and all $t \in T$, we have that $x^{-1}tx = t^i$, where $i = i(x)$ locally.*

(ii) *$T = K_8 \times E$ where K_8 is the quaternion group of order 8 and E is an elementary abelian 2-group. Furthermore, E is central in G and conjugation by a fixed element of G induces in K_8 one of its four inner automorphisms.*

Conversely, if G is a nilpotent or FC group satisfying (i) *or* (ii), *then $TU(\mathbf{Z}G)$ is a subgroup of $U(\mathbf{Z}G)$.*

PROOF. First assume that $TU(\mathbf{Z}G)$ is a subgroup. It follows from the Lemma that $TU(\mathbf{Z}G) = \pm T$, hence [**1**] or [**2**, Theorem II.4.1] shows that T is either abelian or $T = K_8 \times E$ as in (ii).

For elements $t \in T$, and $x \in G$ we consider $u = 1 + (t - 1)x\hat{t}$ where $\hat{t} = 1 + t + \cdots + t^{o(t)-1}$.

Then $u^{-1} = 1 - (t - 1)x\hat{t}$ and

$$utu^{-1} = t + x\hat{t} - 2tx\hat{t} + t^2x\hat{t}.$$

Since $utu^{-1} \in TU(\mathbf{Z}T)$ we must have $txt = xt^i$ or $txt = t^2xt^i$ and consequently $t^x \in \langle t \rangle$.

If T is abelian, the last equality shows that (i) holds. If $T = K_8 \times E$, it readily implies that E is central and that the automorphism induced in K_8 is inner.

Finally assume that either (i) or (ii) holds. From [**2**, Corollary VI.1.24] we know that $\mathbf{Q}G$ contains no nonzero nilpotent elements; hence, every idempotent is central and [**2**, Lemma VI.3.22] shows that $U(\mathbf{Z}G) = U(\mathbf{Z}T) \cdot G$.

Now, we claim that if $u \in U(\mathbf{Z}G)$ is an element of finite order, then $u \in U(\mathbf{Z}T)$.

Assume that $u = vg$ where $v \in U(\mathbf{Z}T)$ and $g \in G$ is an element of infinite order. Since $gv = v'g$ for some $v' \in U(\mathbf{Z}T)$, if $n = o(u)$ we have that $1 = u^n = v''g^n$, with $v'' \in U(\mathbf{Z}T)$. Consequently $g^n \in U(\mathbf{Z}T)$, a contradiction.

We have thus shown that $TU(\mathbf{Z}G) \subset (\mathbf{Z}T)$.

If (i) holds, $U(\mathbf{Z}T)$ is abelian and the result follows trivially.

If (ii) holds, from [**2**, Corollary II.2.5] we have that $U(\mathbf{Z}T) = \pm T$, hence $U(\mathbf{Z}G) = \pm G$ and thus $TU(\mathbf{Z}G) = \pm T$ is a subgroup. $\square$

As a consequence of the Theorem and a result of Sehgal [**2**, Theorem VI.1.20] we have

COROLLARY. *Let G be a nilpotent or FC group such that $TU(\mathbf{Z}G)$ forms a subgroup. Then $\mathbf{Z}G$ contains no nilpotent elements.*

3. An example. We conclude with an example showing that if G is infinite nilpotent or FC then the fact that $TU(\mathbf{Z}G)$ forms a subgroup does not imply that $U(\mathbf{Z}G)$ is either nilpotent or FC.

Consider the group $G = \langle t, x | t^9 = 1, xtx^{-1} = t^7 \rangle$. Denote $(g_1, g_2) = g_1 g_2 g_1^{-1} g_2^{-1}$, $G^{(1)} = \langle (g_1, g_2) | g_1, g_2 \in G \rangle$ and $G^{(n)} = \langle (x, g) | x \in G^{(n-1)}, g \in G \rangle$.

It is easy to see that $G^{(1)} \subset \langle t \rangle$.

Now, if $g = x^k t^m$ we have

$$(g, t^n) = x^k t^n x^{-k} t^{-n} = t^{n(7^k - 1)} = t^{6n(7^{k-1} + 7^{k-2} + \cdots + 1)} \in \langle t^3 \rangle.$$

In a similar way it follows that $G^{(3)} = \langle 1 \rangle$. Thus G is nilpotent.

Also, if $C(g)$ denotes the conjugacy class of an element $g \in G$, it is easy to see that $C(g) \subset G^{(1)} g = \langle t \rangle g$ and is thus finite.

Consequently, G is also an FC group.

Hence, G satisfies the conditions in our theorem and is both nilpotent and FC, but it follows from Theorems VI.3.23 and VI.5.3 in [2] that $U(\mathbf{Z}G)$ is neither nilpotent nor FC.

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