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M-IDEALS, THE STRONG 2-BALL PROPERTY AND SOME RENORMING THEOREMS

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ABSTRACT. Examples are given of M-ideals in Banach spaces which do not possess the strong 2-ball property. This solves a problem of Alfsen and Effros. A previous example is shown to be incorrect. The technique used to construct these examples is then employed to prove negative renorming theorems for Banach spaces. The following is representative: every separable Banach space has an equivalent norm which is strictly convex but not locally uniformly convex.

Introduction. A closed subspace M of a Banach space E is said to have the n-ball property ($n \in \mathbf{N}$) if, whenever $V_1, \ldots, V_n$ are open balls in E with $M \cap V_i$ nonempty for each i, and $\bigcap_{i=1}^n V_i$ nonempty, then M meets $\bigcap_{i=1}^n V_i$. We say that M has the strong n-ball property in E if the corresponding statement is true for any closed balls $B_1, \ldots, B_n$. We say that M is an M-ideal in E if M^0 is an L-summand in E^*. This means that there is a projection $P\colon E^* \to M^0$ such that $\|f\| = \|Pf\| + \|f - Pf\|$ for all $f \in E^*$. These concepts are due to Alfsen and Effros who showed [**1**, Theorems 5.8 and 5.9] that, in real Banach spaces, the following are equivalent:

(i) M is an M-ideal in E.

(ii) M has the n-ball property, for every n.

(iii) M has the 3-ball property in E.

A simple proof of this, which is also valid for complex Banach spaces, has been given by Lima [**5**, Theorem 6.9]. Alfsen and Effros [**1**, p. 126] showed that an M-ideal need not have the strong 3-ball property, and they asked [**1**, p. 170] if every M-ideal has the strong 2-ball property. We settle this problem in the negative, and show that an earlier example [**5**, p. 40] is incorrect.

It has been observed [**11**, Corollary 2.7] that if E is any Banach space, then $K(E, c_0)$, the space of compact operators from E to c_0, is an M-ideal in $B(E, c_0)$, the corresponding space of bounded linear operators. We prove a renorming theorem which shows that there are many Banach spaces E for which $K(E, c_0)$ does not have the strong 2-ball property in $B(E, c_0)$. (To avoid trivialities, we assume that all Banach spaces are infinite dimensional.) For that result, we need the following concept. Let $\{x_1, x_2, x_3, \ldots\} \subset E$ and $\{f_1, f_2, f_3, \ldots\} \subset E^*$. We say that (x_n, f_n) is a Markuševič basis for E if $f_m(x_n) = \delta_{mn}$, the linear span of $\{x_n\}$ is dense in E, and $\{f_n\}$ separates points of E. Ovsepian and Pełczyński [**7**] showed that every separable Banach space has a Markuševič basis with $\|x_n\| \cdot \|f_n\| < 6$ for all n.

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In §2, the existence of Markuševič bases is used to establish the last result stated in the abstract. This suggests that there is no infinite dimensional Banach space for which every equivalent strictly convex norm is already locally uniformly convex. We recall that a Banach space is strictly convex if every norm one vector is an extreme point of the unit ball; locally uniformly convex if the conditions $\|x_n\| \to \|x\| = 1$ and $\|x_n + x\| \to 2$ imply that $x_n \to x$.

We also recall that a Banach space is uniformly convex if the conditions $\|x_n\| = \|y_n\| = 1$ and $\|x_n + y_n\| \to 2$ imply that $x_n - y_n \to 0$. In §1, unless otherwise specified, the scalars may be real or complex. The scalars are assumed real in §2, although the results for separable Banach spaces also hold in the complex case.

1. M-ideals and the strong 2-ball property. We first note that the example given by Lima [5, p. 40], of an M-ideal failing the strong 2-ball property, is incorrect. Let (e_n) be an orthonormal basis for a complex, separable Hilbert space H. Let E be the real Banach space of all selfadjoint linear operators on H, and M the subspace of selfadjoint compact operators. Then M is an M-ideal in E [1, p. 100]. If $k \in M$ and $\|k\| = 1$, it is easy to show that $M \cap B(\frac{1}{2}, \frac{1}{2})$, $M \cap B(k + \frac{1}{2}, \frac{1}{2})$ and $B(\frac{1}{2}, \frac{1}{2}) \cap B(k + \frac{1}{2}, \frac{1}{2})$ are all nonempty. Let us specify k by

$$k(e_{2n-1}) = n^{-1}\left(e_{2n-1} + (n-1)^{1/2}e_{2n}\right)$$

and

$$k(e_{2n}) = n^{-1}\left((n-1)^{1/2}e_{2n-1} - e_{2n}\right).$$

Lima claims that $M \cap B(\frac{1}{2}, \frac{1}{2}) \cap B(k + \frac{1}{2}, \frac{1}{2})$ is empty. However, let us define an operator a on H by $a(e_1) = e_1$, $a(e_2) = 0$, $a(e_3) = (3e_3 + e_4)/4$, $a(e_4) = (e_3 + e_4)/4$, $a(e_5) = (2e_5 + 2^{-1/2}e_6)/3$, $a(e_6) = (2^{-1/2}e_5 + e_6)/3$ and, for $n > 3$,

$$a(e_{2n-1}) = n^{-1/2}e_{2n-1}, \quad a(e_{2n}) = n^{-1/2}e_{2n}.$$

Then $a \in M \cap B(\frac{1}{2}, \frac{1}{2}) \cap B(k + \frac{1}{2}, \frac{1}{2})$.

We have observed [11, Lemma 2.6] that $c_0(E)$ (the space of sequences in E which converge to zero) is always an M-ideal in $l_\infty(E)$ (the space of bounded sequences in E, equipped with the supremum norm). The next result therefore solves the problem of Alfsen and Effros.

THEOREM 1.1. *If E is strictly convex but not uniformly convex, then $c_0(E)$ does not have the strong 2-ball property in $l_\infty(E)$.*

PROOF. Since E is not uniformly convex, there are $f_n, g_n \in E$ and $\varepsilon > 0$ with $\|f_n\| = \|g_n\| = 1$, $\|f_n + g_n\| \to 2$ but $\|f_n - g_n\| > 2\varepsilon$. Clearly $\varepsilon_n = 1 - \|\frac{1}{2}(f_n + g_n)\| \to 0$. Put $x_n = (1 - \varepsilon_n)^{-1}f_n$ and $y_n = -(1 - \varepsilon_n)^{-1}g_n$. Then $x = (x_n)$ and $y = (y_n)$ belong to $l_\infty(E)$. Clearly $(x_n - f_n) \in c_0(E) \cap B(x, 1)$ and $(y_n + g_n) \in c_0(E) \cap B(y, 1)$. Since $\|x_n - y_n\| = 2$ for all n, it follows from the strict convexity of E that the only point in $B(x, 1) \cap B(y, 1)$ is $z = (\frac{1}{2}(x_n + y_n))$. But $\|\frac{1}{2}(x_n + y_n)\| > \varepsilon$, so $z \notin c_0(E)$.

There are many Banach spaces satisfying the hypotheses of Theorem 1.1. Theorem 2.1 and Corollary 2.2 will show that if E is real and separable, weakly compactly generated or $L_1(\mu)$ for some measure μ, then E is renormable so as to be strictly convex but not locally uniformly convex.

Now $K(E, c_0)$ may be identified with $c_0(E^*)$, and $B(E, c_0)$ with $\{(f_n) \in l_\infty(E^*): f_n \to 0 \text{ weak}^*\}$. Thus $K(E, c_0)$ is always an M-ideal in $B(E, c_0)$. If E^* is strictly convex, and $f_n, g_n \in E^*$ from the previous proof can be chosen so that $f_n, g_n \to 0$ weak*, then $K(E, c_0)$ will not have the strong 2-ball property in $B(E, c_0)$. This is often the case.

THEOREM 1.2. *Let E be any separable Banach space. Then E can be renormed so that $K(E, c_0)$ does not have the strong 2-ball property in $B(E, c_0)$.*

PROOF. Let (x_n, h_n) be a Markuševič basis for E with $\|h_n\| = 1$, and $\{x_n\}$ bounded. Define an equivalent norm, $|||\cdot|||$, on E^* by

$$|||f||| = \left\{ \sum_{n=1}^{\infty} 4^{-n}|f(x_n)|^2 \right\}^{1/2} + \max\left\{ \|f\|, \sup_{n=1}^{\infty} (|f(x_n)| + |f(x_{n+1})|) \right\}.$$

Then $|||\cdot|||$ is weak* lower semicontinuous, and so arises from an equivalent norm on E. The first term in the expression above ensures that $(E^*, |||\cdot|||)$ is strictly convex. Now $|||h_n||| = 1 + 2^{-n} \to 1$, $|||h_n \pm h_{n+1}||| \to 2$ and $h_n \to 0$ weak*. Put $f_n = (1 + 2^{-n})^{-1}h_n$, $g_n = f_{n+1}$ and proceed as in Theorem 1.1.

We remark that if E is separable but not reflexive, then there is an easier proof of Theorem 1.2. Suppose E^* is strictly convex, and choose $\psi \in E^{***}$ with $\|\psi\| = 1$ and $\psi(\hat{E}) = \{0\}$. (We use ˆ to denote the canonical embeddings of E into E^{**} and of E^* into E^{***}.) By Goldstine's theorem, there is a net (h_α) in E^* such that $\hat{h}_\alpha \to \psi$ weak*in E^{***}, and $\|h_\alpha\| = 1$. Note that $h_\alpha \to 0$ weak* in E^*. Now $\hat{h}_\beta + \hat{h}_\gamma \to 2\psi$ weak*, so $\|h_\beta + h_\gamma\| \to 2$. Clealy (h_α) is not a Cauchy net, so there is an $\varepsilon > 0$ such that $(\forall\alpha)(\exists\beta, \gamma > \alpha)\|h_\beta - h_\gamma\| > \varepsilon$. Put $f_\alpha = h_\beta$ and $g_\alpha = h_\gamma$. Then (f_α), (g_α) are nets with the properties required in the preceding proof. Since the unit ball of E^* is weak metrizable, sequences with the required properties are easily constructed.

2. Further renorming theorems. It is clear that every locally uniformly convex Banach space is strictly convex, and examples are known which show that the converse is false. We show that the converse fails in a very strong sense.

THEOREM 2.1. *Let E be any separable Banach space. Then E has an equivalent norm which is strictly convex but not locally uniformly convex.*

PROOF. Let (x_n, f_n) be a Markuševič basis for E, with $\|x_n\| = 1$ and $\{f_n\}$ bounded. Define $|||\cdot|||$ on E by

$$|||x||| = \left\{ \sum_{n=1}^{\infty} 4^{-n}|f_n(x)|^2 \right\}^{1/2} + \max\left\{ \frac{1}{2}\|x\|, \frac{1}{2}|f_1(x)| + \sup_{n=2}^{\infty} |f_n(x)| \right\}.$$

Then $|||\cdot|||$ is an equivalent strictly convex norm for E. Note that $|||x_1||| = 1$, $|||x_n||| \to 1$ and $|||x_1 + x_n||| \to 2$. However $f_1(x_n) \to 0$ and $f_1(x_1) = 1$, so (x_n) does not even converge weakly to x_1.

Troyanski [10] showed that a Banach space has an equivalent locally uniformly convex norm if it is $L_1(\mu)$ for some measure μ, or if it is weakly compactly generated (i.e. if it contains a weakly compact set whose linear span is dense). The conclusion of Theorem 2.1 holds for all such spaces.

COROLLARY 2.2. *Suppose that $E = L_1(\mu)$, or that E is weakly compactly generated. Then E has an equivalent norm which is strictly convex but not locally uniformly convex.*

PROOF. It is easy to show that $L_1(\mu)$ has a complemented subspace isometric to l_1. According to [2, Lemma 4], any weakly compactly generated Banach space contains a complemented separable subspace. In either case, we may write $E = M \oplus N$, where M is separable. Let $\|\cdot\|_1$ be a norm on M as given by Theorem 2.1, and let $\|\cdot\|_2$ be a strictly convex norm for E. For $x \in M, y \in N$, put $\|x + y\| = (\|x\|_1^2 + \|y\|_2^2)^{1/2}$. Then $\|\cdot\|$ has the required properties.

This technique can be used to prove a multitude of results similar to Theorem 2.1. We will give only one more. Lovaglia [6] showed that if E^* is locally uniformly convex, then the norm on E is Fréchet differentiable. The next theorem, which is inspired by [8, Example 3], shows that the dual result is far from true. First we recall that Cudia [3, Corollary 3.18] showed that the norm on E^* is Fréchet differentiable if and only if E has the property that $x_n - y_n \to 0$ whenever $\|x_n\| = \|y_n\| = 1$ and $f(x_n + y_n) \to 2\|f\|$ for some nonzero f in E^*.

THEOREM 2.3. *Let E be a reflexive Banach space. Then E can be renormed so that E^* has Fréchet differentiable norm, but E is not locally uniformly convex.*

PROOF. Arguing as in Corollary 2.2, we may suppose that E is separable. By [4] we may assume that $\|\cdot\|$, the original norm on E, is strictly convex, and has the Radon-Riesz property (i.e. $y_n \to y$ whenever $y_n \to y$ weakly and $\|y_n\| \to \|y\|$). Now let (x_n, f_n) be a Markuševič basis for E with $\|x_n\| = 1$ and $\{f_n\}$ bounded. Define $|||\cdot|||$ on E by

$$|||x|||^2 = (|f_1(x)| + \|x - f_1(x)x_1\|)^2 + \sum_{n=2}^{\infty} 2^{-n}|f_n(x)|^2.$$

Then $|||x_n||| \to |||x_1||| = 1$ and $|||x_1 + x_n||| \to 2$ but $x_n \not\to x_1$. So $|||\cdot|||$ is not locally uniformly convex.

Since $\bigcap_{n=2}^{\infty} \ker f_n$ meets the hyperplane $\ker f_1$ in $\{0\}$, we see that $\bigcap_{n=2}^{\infty} \ker f_n$ is at most one dimensional. Hence $\bigcap_{n=2}^{\infty} \ker f_n$ is just the linear span of x_1. Using this, and the fact that $\|\cdot\|$ is strictly convex, a routine argument shows that $|||\cdot|||$ is strictly convex. Since $\|\cdot\|$ has the Radon-Riesz property, it is easy to show that $|||\cdot|||$ also enjoys the Radon-Riesz property. Finally, suppose $y_n, z_n \in E, f \in E^*$ and that $|||y_n||| = |||z_n||| = |||f||| = 1, f(y_n + z_n) \to 2$. Since E is reflexive, a subsequence argument enables us to assume that $y_n \to y$ weakly and $z_n \to z$ weakly $(y, z \in E)$. Then $f(y + z) = 2$, $|||y|||, |||z||| \leq 1$. It follows from strict convexity that $y = z$, $|||y||| = 1$. Now the Radon-Riesz property tells us that $y_n \to y$ and $z_n \to z$. Thus $y_n - z_n \to 0$, so E^* is Fréchet differentiable under $|||\cdot|||$.

The hypothesis of reflexivity is essential in Theorem 2.3 since Smulian [9] has shown that E must be reflexive if E^* has Fréchet differentiable norm.

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