

STABILITY OF PERIODIC ORBITS IN THE THEOREM OF ŠARKOVSKII

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ABSTRACT. Let f be a continuous map of a closed, bounded interval into itself. It is shown that the conclusion of the theorem of Šarkovskii holds for perturbations of f . In other words, if f has a periodic point of period k , and g is a continuous map close to f , then g has periodic points of certain periods.

1. Introduction. This paper is concerned with the periodic orbits of continuous mappings of the interval to itself. Such mappings (sometimes called first order difference equations when studied from this point of view) arise as models for various phenomena in the natural sciences. See [2] or [3] for some examples and references.

Let I denote a closed, bounded, interval on the real line, and let $C^0(I, I)$ denote the space of continuous mappings of I into itself with the topology of uniform convergence. For $f \in C^0(I, I)$ let $P(f)$ denote the set of positive integers k such that f has a periodic point of least period k .

Consider the following ordering of the positive integers:

$$1, 2, 4, 8, \dots \dots 7 \cdot 8, 5 \cdot 8, 3 \cdot 8, \dots, \\ 7 \cdot 4, 5 \cdot 4, 3 \cdot 4, \dots, 7 \cdot 2, 5 \cdot 2, 3 \cdot 2, \dots, 7, 5, 3.$$

We will refer to this ordering as the Šarkovskii ordering. A. N. Šarkovskii has proved the following theorem (see [4] or [5]).

THEOREM (ŠARKOVSKII). *Let $f \in C^0(I, I)$. If $n \in P(f)$ and k is to the left of n in the Šarkovskii ordering then $k \in P(f)$.*

In studying properties of the orbits of mappings, it is desirable to prove that a given property is stable, in the sense that if a map f has the property then perturbations of f (maps close to f in an appropriate topological space) also have the given property. This is important from a physical point of view, because when one uses a mapping as a model of some phenomena there will normally be some possibility for error. For most stability theorems, one must consider C^1 mappings with some restrictions and C^1 perturbations. However, in this paper we prove the following result, which shows that some stability exists in Šarkovskii's Theorem, in the C^0 topology for all $f \in C^0(I, I)$.

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MAIN THEOREM. *Let $f \in C^0(I, I)$ and suppose that $n \in P(f)$. There is a neighborhood N of f in $C^0(I, I)$ such that for every $g \in N$ and every positive integer k with k to the left of n in the Šarkovskii ordering, $k \in P(g)$.*

In proving this result, we use a theorem of Štefan, [5], and a lemma from [1] (both of which will be stated in the next section), in addition to the theorem of Šarkovskii.

We remark that in general if $f \in C^0(I, I)$ (or $f \in C^1(I, I)$) and $n \in P(f)$ then perturbations of f need not have a periodic point of period n .

2. Preliminary definitions and results. Let $f \in C^0(I, I)$. For any nonnegative integer n , we define f^n inductively by f^0 is the identity map of I and $f^n = f \circ f^{n-1}$.

Let $p \in I$. We say p is a periodic point of f if p is a fixed point of f^n for some positive integer n . If p is a periodic point of f , the smallest positive integer n with $f^n(p) = p$ is called the period of p .

We define the orbit of p to be $\{f^n(p): n = 0, 1, 2, \dots\}$. If p is a periodic point, we say the orbit of p is a periodic orbit, and we define the period of the orbit to be the period of p .

Finally, for $f \in C^0(I, I)$ we let $P(f)$ denote the set of positive integers k such that f has a periodic point of period k .

We now state two elementary lemmas which will be used in the next section.

LEMMA 1. *Let $f \in C^0(I, I)$. Suppose M and L are closed subintervals of I such that $f(M) \supset L$. Then there is a closed interval $K \subset M$ with $f(K) = L$.*

LEMMA 2. *Let $f \in C^0(I, I)$. Suppose K is a closed subinterval of I with $K \subset f(K)$. Then f has a fixed point in K .*

We will also use the following lemma from [1] (see Lemma 10 in §3 of [1]).

LEMMA 3. *Let $f \in C^0(I, I)$. Suppose f has a periodic orbit $\{p_1, p_2, p_3, p_4\}$ of period 4 with $p_1 < p_2 < p_3 < p_4$. If $f(\{p_1, p_2\}) \neq \{p_3, p_4\}$ then f has a periodic point of period 3.*

We conclude this section by stating the following theorem of Štefan (see Theorem 2 of [5]) which will be used in the next section.

THEOREM (ŠTEFAN). *Let $f \in C^0(I, I)$. Suppose $n \in P(f)$ where n is odd and $n > 1$, but $j \notin P(f)$ for all $j \in \{3, 5, \dots, n-2\}$. Let $\{p_1, \dots, p_n\}$ be a periodic orbit of f of period n with $p_1 < p_2 < \dots < p_n$. Let $t = (n+1)/2$. Then either (A) or (B) holds.*

(A) $f(p_{t-k}) = p_{t+k}$ for $k = 1, \dots, t-1$, $f(p_{t+k}) = p_{t-k-1}$ for $k = 0, \dots, t-2$, and $f(p_n) = p_t$.

(B) $f(p_{t-k}) = p_{t+k+1}$ for $k = 0, \dots, t-2$, $f(p_{t+k}) = p_{t-k}$ for $k = 1, \dots, t-1$, and $f(p_1) = p_t$.

3. Proof of the Main Theorem.

LEMMA 4. *Let $f \in C^0(I, I)$ and let k be an odd positive integer with $k > 3$. Suppose that there is a point $y \in I$ such that the following hold.*

- (1) $f^{k-2}(y) < f^{k-4}(y) < f^{k-6}(y) < \dots < f^3(y) < f(y) < y$.
- (2) $y < f^2(y) < f^4(y) < f^6(y) < \dots < f^{k-1}(y)$.
- (3) $y < f^k(y)$.

Then $k \in P(f)$.

PROOF. Since $f(y) < y$ and $f(f(y)) > f(y)$, f has a fixed point $e \in (f(y), y)$. Let $M_1 = [e, y]$, $M_3 = [y, f^2(y)]$, \dots , and $M_k = [f^{k-3}(y), f^{k-1}(y)]$. Also, let $M_2 = [f(y), e]$, $M_4 = [f^3(y), f(y)]$, \dots , and $M_{k-1} = [f^{k-2}(y), f^{k-4}(y)]$.

Note that $f(M_j) \supset M_{j+1}$ for each $j = 1, \dots, k - 1$ and $f(M_k) \supset M_1$. Hence, by Lemma 1, there are closed intervals K_1, K_2, \dots, K_k with $K_j \subset M_j$ for each $j = 1, \dots, k$ such that $f(K_k) = M_1$ and $f(K_j) = K_{j+1}$ for each $j = 1, \dots, k - 1$. It follows that $f^k(K_1) = M_1$. By Lemma 2, f^k has a fixed point $z \in K_1$.

Suppose that $f^j(z) = z$ for some $j \in \{1, \dots, k - 1\}$. Since $f^j(z) \in M_{j+1}$, it follows that either $j = 1$ and $z = e$, or $j = 2$ and $z = y$. But $z = e$ is impossible since $f^2(z) \in M_3$. Also, $j = 2$ and $z = y$ is impossible since $y < f^2(y)$. Hence $f^j(z) \neq z$ for all $j \in \{1, \dots, k - 1\}$. Thus, z is a periodic point of f of period k .

LEMMA 5. Let $f \in C^0(I, I)$ and suppose $n \in P(f)$ where n is odd and $n \geq 3$. Then there is a neighborhood N of f in $C^0(I, I)$ with the property that if $g \in N$ then $(n + 2) \in P(g)$.

PROOF. By the theorem of Šarkovskii it suffices to prove this lemma in the case that $j \notin P(f)$ for all $j \in \{3, 5, \dots, n - 2\}$. Hence, we assume that this is the case.

Since $n \in P(f)$, f has a periodic orbit $\{p_1, \dots, p_n\}$ of period n , with $p_1 < p_2 < \dots < p_n$. Either (A) or (B) (of the theorem of Štefan) must hold. Since these are analogous we may assume that (A) holds.

Let $t = (n + 1)/2$ and let $z = p_t$. Then

- (1) $f^{n-2}(z) < f^{n-4}(z) < f^{n-6}(z) < \dots < f^3(z) < f(z) < z$.
- (2) $z < f^2(z) < f^4(z) < f^6(z) < \dots < f^{n-1}(z)$.
- (3) $z = f^n(z)$.

Now, since $f(f(z)) > z$ and $f(z) < z$, there is a point $b \in (f(z), z)$ with $f(b) = z$. Also, since $f(b) > b$ and $f(z) < b$, there is a point $y \in (b, z)$ with $f(y) = b$. Thus, $f^2(y) = z$. It follows that y satisfies the conditions of Lemma 4 with $k = n + 2$. Thus there is a neighborhood N of f such that y satisfies the conditions of Lemma 4 with $k = n + 2$ for all $g \in N$. Therefore $n + 2 \in P(g)$ for all $g \in N$.

LEMMA 6. Let $f \in C^0(I, I)$. Suppose $n \in P(f)$ and $n = r \cdot s$ where $r = 2^i$ for some nonnegative integer i , and s is an odd integer with $s \geq 3$. Then there is a neighborhood N of f in $C^0(I, I)$ such that for every $g \in N$, and every positive integer k with k to the left of n in the Šarkovskii ordering, $k \in P(g)$.

PROOF. Note that $n \in P(f)$ implies that $s \in P(f')$. By Lemma 5, there is a neighborhood N_1 of f' with the property that if $g \in N_1$ then $(s + 2) \in P(g)$.

Since the mapping g to g' is continuous, there is a neighborhood N of f such that if $g \in N$ then $g' \in N_1$.

Let $g \in N$. Then $g' \in N_1$. Hence, $(s + 2) \in P(g')$. This implies that $(s + 2) \cdot j \in P(g)$, where $j = 2^t$ for some integer t with $0 \leq t \leq i$. It follows from the theorem

of Šarkovskii that for every positive integer k with k to the left of n in the Šarkovskii ordering, $k \in P(g)$.

LEMMA 7. Let $f \in C^0(I, I)$ and suppose $4 \in P(f)$. Then there is a neighborhood N of f in $C^0(I, I)$ such that if $g \in N$ then $2 \in P(g)$.

PROOF. By hypothesis f has a periodic orbit $\{p_1, p_2, p_3, p_4\}$ of period 4 with $p_1 < p_2 < p_3 < p_4$. If $3 \in P(f)$ then the conclusion of this lemma follows from Lemma 6. Hence, we may assume that $3 \notin P(f)$. This implies, by Lemma 3, that $f(\{p_1, p_2\}) = \{p_3, p_4\}$. Hence, we must also have $f(\{p_3, p_4\}) = \{p_1, p_2\}$. Thus, $f([p_1, p_2]) \supset [p_3, p_4]$ and $f([p_3, p_4]) \supset [p_1, p_2]$.

By Lemma 1, there is a closed interval $K \subset [p_3, p_4]$ with $f(K) = [p_1, p_2]$. Let v be the element of $\{p_1, p_2\}$ with $f(v) = p_4$. For some $x \in K$, $f(x) = v$. Since $x \neq p_4$, $f^2(x) > x$. Similarly $f^2(y) < y$ for some $y \in K$. There is a neighborhood N of f such that if $g \in N$, then $g^2(x) > x$, $g^2(y) < y$ and $g(z) < p_3$ for all $z \in K$. Then $2 \in P(g)$ for all $g \in N$.

LEMMA 8. Let $f \in C^0(I, I)$. Suppose $2^m \in P(f)$ for some positive integer m . Then there is a neighborhood N of f in $C^0(I, I)$ such that if $g \in N$ then $2^i \in P(g)$ for every integer i with $0 < i \leq m - 1$.

PROOF. We may assume $m \geq 2$. Let $r = 2^{m-2}$. Then $4 \in P(f^r)$. By Lemma 7, $2 \in P(g)$ for every g in some neighborhood N_1 of f^r . There is a neighborhood N of f such that if $g \in N$ then $g^r \in N_1$.

Let $g \in N$. Then since $2 \in P(g^r)$, $2^{m-1} \in P(g)$. By the theorem of Šarkovskii, it follows that $2^i \in P(g)$ for every integer i with $0 < i < m - 1$.

The Main Theorem follows immediately from Lemma 6 and Lemma 8.

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