PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 81, Number 2, February 1981

THE CENTER OF A CONVEX SET

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Let X be a Banach space and K a weakly compact convex nonvoid subset with normal structure [1]. Brodskii and Mil'man [1] constructed, using transfinite induction, a "center" of K which is fixed by every isometry mapping K *onto* K. In this note, we construct a unique "center" for a weakly compact convex nonvoid subset (not necessarily having normal structure) which is fixed by every *affine* isometry mapping K *into* K. A similar theorem for weak* compact convex sets is also possible under some additional assumptions.

CONSTRUCTION. Let K be a nonempty weakly compact convex subset of a Banach space. We shall define C_α for all ordinals α by transfinite induction. Set $C_0 = K$. Let β be an ordinal and suppose that C_α has been defined for $\alpha < \beta$ in such a way that (i) each C_α is a nonempty closed convex subset of K and (ii) C_α, $\alpha < \beta$, is decreasing. If β is a limit ordinal, we set $C_\beta = \bigcap_{\alpha<\beta} C_\alpha$. Otherwise, let γ be the predecessor of β and let

$$S_\beta = \left\{ z \in C_\gamma : z = \tfrac{1}{2}(x + y) \text{ for some } x, y \in C_\gamma \text{ with } \|x - y\| = \tfrac{1}{2} \operatorname{diam} C_\gamma \right\}.$$

Then we set $C_\beta = \overline{\mathrm{Co}}\, S_\beta$. Since C_γ is the closed convex hull of its strongly exposed points (see [2]), it is easy to see that if card $C_\gamma > 1$, C_β contains no strongly exposed points of C_γ and hence is a *proper* subset of C_γ. If card $C_\gamma = 1$, $C_\beta = C_\gamma$. It follows that for sufficiently large ordinals δ, C_δ are identical and consist of exactly one point which we call the center of K.

If X is a Banach space such that the dual of every separable subspace of X is separable, and K is a nonempty weak* compact convex subset of X^*, then every weak* closed convex nonempty subset of K is the weak* closed convex hull of its weak* strongly exposed points (see [5]-[8]). With appropriate changes, the prior construction applies to this situation; in particular, replacing C_β by $\overline{\mathrm{Co}}^*(S_\beta)$, where $^{-*}$ denotes the weak* closure. Thus K has a unique center.

THEOREM 1. *Let K be a nonempty weakly compact convex subset of a Banach space. The center of K is a fixed point of every affine isometry mapping K into K.*

PROOF. Note that in the construction, each C_α is mapped into itself by every affine isometry of K into K.

Received by the editors April 21, 1980.
1980 *Mathematics Subject Classification*. Primary 46B20; Secondary 47H10.

THEOREM 2. *Let X be a Banach space such that the dual of every separable subspace of X is separable. Let K be a weak* compact convex nonempty subset of X^*. The center of K is a fixed point of every weak* continuous affine isometry mapping K into K.*

PROOF. If T is an affine isometry, then $T(\text{Co } S_\beta) \subseteq \text{Co } S_\beta$. By the weak* continuity, $T(C_\beta) = T(\overline{\text{Co}}^* S_\beta) \subseteq C_\beta$.

REMARKS. 1. It also follows from the Ryll-Nardzewski fixed point theorem (see **[4]**) that the family of affine isometries on K has a common fixed point (which is not necessarily the center). Our approach follows that of Namioka-Asplund **[4]**.

2. The assumption of weak* continuity in Theorem 2 cannot be removed since Example 1 in **[3]** shows that there are fixed point free affine isometries.

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