PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 81, Number 3, March 1981

ON NEAR-DERIVATIONS

GYULA MAKSA

ABSTRACT. In this note we show how near-derivations can be expressed by biadditive and additive functions satisfying further conditions.

1. The concept of near-derivation has been introduced and discussed by Lawrence, Mess and Zorzitto [3] in connection with nonnegative information functions. A real-valued function γ defined on the reals $\mathbf{R}$ is called a near-derivation if

$$\gamma(xy) = x\gamma(y) + y\gamma(x) \quad \text{for all } x, y \in \mathbf{R}, \tag{1}$$

$$\gamma(x + y) \geqslant \gamma(x) + \gamma(y) \quad \text{for all } x \geqslant 0, y \geqslant 0, \tag{2}$$

$$\gamma(r) = 0 \quad \text{for all rational } r. \tag{3}$$

Daróczy and Maksa showed in [1] that there exists a near-derivation which is not a derivation. Namely, if d: $\mathbf{R} \to \mathbf{R}$ is a nonidentically zero derivation then the function γ defined by

$$\gamma(x) = \begin{cases} d(d(x)) - d(x)^2/x & \text{if } x \neq 0, \\ 0 & \text{if } x = 0 \end{cases} \tag{4}$$

is such a near-derivation. In this note we present a method for recovering near-derivations in terms of biadditive functions on $\mathbf{R}^2$ and additive functions on $\mathbf{R}$ satisfying further conditions.

2. It has been proved in [3] that for any near-derivation γ the finite limit

$$\alpha(x) = \lim_{n\to\infty} \gamma(x + n) \tag{5}$$

exists for all $x \in \mathbf{R}$. Furthermore, the function α: $\mathbf{R} \to \mathbf{R}$ defined by (5) has the following properties:

(a) $\gamma(x) \leqslant \alpha(x)$ for all $x \geqslant 0$,

(b) $\alpha(x + y) = \alpha(x) + \alpha(y)$ for all $x, y \in \mathbf{R}$,

(c) $2x\alpha(x) \leqslant \alpha(x^2)$ for all $x \in \mathbf{R}$,

(d) $\alpha(r) = 0$ for all rational r.

Using (a) and (1)–(3) we have

$$t\alpha\left(\frac{1}{t}\right) + \frac{1}{t}\alpha(t) = |t|\alpha\left(\frac{1}{|t|}\right) + \frac{1}{|t|}\alpha(|t|)$$
$$\geqslant |t|\gamma\left(\frac{1}{|t|}\right) + \frac{1}{|t|}\gamma(|t|) = \gamma(1) = 0$$

Received by the editors July 6, 1979 and, in revised form, February 29, 1980.
1980 *Mathematics Subject Classification*. Primary 39B99.

that is,

(e) $t\alpha(1/t) + (1/t)\alpha(t) \geqslant 0$ for all $t \in \mathbf{R} \setminus \{0\}$.

Suppose that the function α: $\mathbf{R} \to \mathbf{R}$ satisfies (b)–(e) and define A: $\mathbf{R}^2 \to \mathbf{R}$ by

$$A(x,y) = \alpha(xy) - x\alpha(y) - y\alpha(x). \tag{6}$$

It is easy to see that A has the properties:

$$A(x,y) = A(y,x), \tag{7}$$

$$A(x+y,z) = A(x,z) + A(y,z), \tag{8}$$

$$A(x,x) \geqslant 0, \tag{9}$$

$$A(xy,z) + zA(x,y) = A(x,yz) + xA(y,z), \tag{10}$$

$$A(t,1/t) \leqslant 0, \tag{11}$$

for all $x, y, z \in \mathbf{R}$ and $t \in \mathbf{R} \setminus \{0\}$. It follows from [2] that a function A: $\mathbf{R}^2 \to \mathbf{R}$ satisfying (7)–(11) is always of the form (6) where α: $\mathbf{R} \to \mathbf{R}$ has the properties (b)–(e).

THEOREM 1. *Suppose that the function* α: $\mathbf{R} \to \mathbf{R}$ *satisfies* (b)–(e) *and define* A: $\mathbf{R}^2 \to \mathbf{R}$ *by* (6). *Then the function* γ *given by*

$$\gamma(x) = \begin{cases} \alpha(x) - \sum_{n=1}^{\infty} 2^{n-1}x^{1-1/2^{n-1}}A(x^{1/2^n}, x^{1/2^n}) & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -\gamma(-x) & \text{if } x < 0 \end{cases} \tag{12}$$

is a near-derivation.

PROOF. Let $\gamma_n(x) = 2^n x^{1-1/2^n}\alpha(x^{1/2^n})$ for $x > 0$ and $n = 0, 1, \ldots$. Then

$$\gamma_n(x) - \gamma_{n-1}(x) = -2^{n-1}x^{1-1/2^{n-1}}A(x^{1/2^n}, x^{1/2^n}). \tag{13}$$

Hence, by (9), we get that the sequence $(\gamma_n(x))$ is decreasing for all fixed $x > 0$. According to (c) and (11)

$$\gamma_n(x) + x^2\gamma_{n-1}(1/x) \geqslant -2^n xA(x^{1/2^n}, x^{-1/2^n}) \geqslant 0$$

and therefore

$$\gamma_n(x) \geqslant -x^2\gamma_{n-1}(1/x) \geqslant -x^2\gamma_0(1/x),$$

which means that the sequence $(\gamma_n(x))$ is bounded below for all fixed $x > 0$. Thus $(\gamma_n(x))$ is convergent and its limit is in $\mathbf{R}$. On the other hand (13) and (12) imply that

$$\gamma(x) = \lim_{n\to\infty} \gamma_n(x) \tag{14}$$

holds for all $x > 0$. By the Cauchy-Schwarz inequality for bilinear forms on a rational vector space

$$|A(u,v)| \leqslant \sqrt{A(u,u)}\,\sqrt{A(v,v)} \tag{15}$$

for all $u, v \in \mathbf{R}$. Let $x > 0$, $y > 0$ and $u = x^{1/2^n}$, $v = y^{1/2^n}$ $(n = 0, 1, \ldots)$. Using (6), the definition of $(\gamma_n(x))$ and (13), (15) implies that

$$|\gamma_n(xy) - x\gamma_n(y) - y\gamma_n(x)| \leqslant 2\sqrt{xy}\,\sqrt{\gamma_{n-1}(x) - \gamma_n(x)}\,\sqrt{\gamma_{n-1}(y) - \gamma_n(y)}.$$

Hence, by (14), we obtain (1) for all $x > 0, y > 0$. From (12) and (9) $\gamma(t) \leq \alpha(t)$ for $t > 0$. Thus

$$\gamma\left(\frac{x}{x+y}\right) + \gamma\left(\frac{y}{x+y}\right) \leq \alpha\left(\frac{x}{x+y}\right) + \alpha\left(\frac{y}{x+y}\right) = \alpha(1) = 0$$

for all $x > 0, y > 0$. By (1) this implies (2) for $x > 0, y > 0$. To prove (3) let r be a positive rational number. Then $A(r,r) = 0$ therefore by (15) $A(r,u) = 0$ for all $u \in \mathbf{R}$. Substituting $x = y = \sqrt{r}$, $z = 1/\sqrt{r}$ in (10) we see that $A(\sqrt{r}, \sqrt{r}) = rA(\sqrt{r}, 1/\sqrt{r})$. Thus (9) and (11) give that $A(\sqrt{r}, \sqrt{r}) = 0$. By induction we have $A(r^{1/2^n}, r^{1/2^n}) = 0$; thus (12) and (d) imply (3) for all positive rational r. Since γ is an odd function the proof is complete.

THEOREM 2. *Let γ be a near-derivation. Then there exist functions α: $\mathbf{R} \to \mathbf{R}$ and A: $\mathbf{R}^2 \to \mathbf{R}$ satisfying* (b) *and* (7)–(11), *respectively such that* (12) *holds for all $x \in \mathbf{R}$.*

PROOF. We have known that there exists a function α: $\mathbf{R} \to \mathbf{R}$ with the properties (a)–(e). Thus the function A: $\mathbf{R}^2 \to \mathbf{R}$ given by (6) satisfies (7)–(11). Define the function δ on $\mathbf{R}$ by

$$\delta(x) = \begin{cases} \alpha(x) - \sum_{n=1}^{\infty} 2^{n-1} x^{1-1/2^{n-1}} A(x^{1/2^n}, x^{1/2^n}) & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -\delta(-x) & \text{if } x < 0. \end{cases}$$

Applying Theorem 1 we get that δ is a near-derivation. Using (6), (a) and (1) we have for $x > 0$

$$\delta(x) = \lim_{n\to\infty} 2^n x^{1-1/2^n} \alpha(x^{1/2^n}) \geq \lim_{n\to\infty} 2^n x^{1-1/2^n} \gamma(x^{1/2^n}) = \gamma(x).$$

Since $\delta - \gamma$ satisfies (1) this implies that $\delta = \gamma$, thus the proof is complete.

We remark that if d is a nonidentically zero derivation and $\alpha(x) = d(d(x))$, $A(x,y) = 2d(x)d(y)$ $(x, y \in \mathbf{R})$ then Theorem 1 gives the example (4).

REFERENCES

1. Z. Daróczy and Gy. Maksa, *Nonnegative information functions*, Analytic Function Methods in Probability and Statistics, Colloq. Math. Soc. J. Bolyai **21** (1979), 65–76.

2. B. Jessen, J. Karpf and A. Thorup, *Some functional equations in groups and rings*, Math. Scand. **22** (1968), 257–265.

3. J. Lawrence, G. Mess and F. Zorzitto, *Near-derivations and information functions*, Proc. Amer. Math. Soc. **76** (1979), 117–122.

DEPARTMENT OF MATHEMATICS, KOSSUTH UNIVERSITY, H-4010 DEBRECEN, PF 12, HUNGARY