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EGOROFF'S THEOREM AND THE DISTRIBUTION OF STANDARD POINTS IN A NONSTANDARD MODEL[1]

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ABSTRACT. We study the relationship between the Loeb measure ${}^0({}^*\mu)$ of a set E and the μ-measure of the set $S(E) = \{x | {}^*x \in E\}$ of standard points in E. If E is in the σ-algebra generated by the standard sets, then ${}^0({}^*\mu)(E) = \mu(S(E))$. This is used to give a short nonstandard proof of Egoroff's Theorem. If E is an internal, * measurable set, then in general there is no relationship between the measures of $S(E)$ and E. However, if ${}^*X$ is an ultrapower constructed using a minimal ultrafilter on ω, then ${}^*\mu(E) \approx 0$ implies that $S(E)$ is a μ-null set. If, in addition, μ is a Borel measure on a compact metric space and E is a Loeb measurable set, then

$$\underline{\mu}(S(E)) < {}^0({}^*\mu)(E) < \bar{\mu}(S(E))$$

where $\underline{\mu}$ and $\bar{\mu}$ are the inner and outer measures for μ.

The work in this paper was originally stimulated by the search for an illuminating nonstandard proof of Egoroff's Theorem. Despite the importance of such a proof it has been surprisingly elusive (see, for example, [8] or [11]). §I of this paper presents a short, natural proof of Egoroff's Theorem using a result from §II on the distribution of standard points in a nonstandard model. The work in §II is of independent interest.

Throughout this paper $(X, \mathfrak{F}, \mu)$ will denote a (standard) positive measure space with $\mu(X)$ finite; $\mathfrak{M}$ will denote a standard higher order model of X along with the real numbers, R; and ${}^*\mathfrak{M}$ will denote a proper nonstandard extension of $\mathfrak{M}$. We will always assume ${}^*\mathfrak{M}$ is $\aleph_1$-saturated, but any further assumptions will be explicitly stated. If P is an entity in $\mathfrak{M}$, ${}^*P$ will denote the corresponding entity in ${}^*\mathfrak{M}$. Thus, in particular, ${}^*\mu$: ${}^*\mathfrak{F} \to {}^*[0, \infty)$ denotes the extension in ${}^*\mathfrak{M}$ of μ to the * measurable sets. We use the usual notation $\mathrm{St}(x)$ for the standard part of a finite nonstandard real and $x \approx y$ for x infinitely close to y.

I. Egoroff's Theorem. Suppose $f_1, f_2, \ldots$ is a standard sequence of measurable functions $X \to R$ and f: $X \to R$ is also a measurable function. Egoroff's Theorem [3] states

I.1 EGOROFF'S THEOREM. *If $f_n \to f$ pointwise almost everywhere then for every $\varepsilon > 0$ there is a set $A \in \mathfrak{F}$ such that $\mu(A) < \varepsilon$ and $f_n \to f$ uniformly on $X \setminus A$.*

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If $\{f_n\}$ satisfies the conclusion of Egoroff's Theorem we will say $\{f_n\}$ converges to f *nearly uniformly*. Note that Egoroff's Theorem is false without the assumption that $\mu(X)$ is finite.

The following characterization of nearly uniform convergence is essentially due to Robinson [**8**].

I.2 DEFINITION. Suppose $f, f_1, f_2, \ldots$ are standard measurable functions $X \to R$, and $x \in {}^*X$. Then x is said to be a *point of intrinsic nonuniformity* if there is an infinite integer ν such that $f_\nu(x) \not\approx {}^*f(x)$. Let E denote the set of points of intrinsic nonuniformity. (Note: E is usually external.)

I.3 DEFINITION. Suppose A is a (possibly external) subset of ${}^*X$. A is said to have *S-measure zero* if for every standard $\varepsilon > 0$ there is a standard set $B \in \mathfrak{F}$ such that $A \subseteq {}^*B$ and $\mu(B) < \varepsilon$.

I.4 PROPOSITION. *Using the notation of Definition* I.2, *the following are equivalent.*

(i) $\{f_n\}$ *converges to* f *nearly uniformly.*

(ii) E *has S-measure zero.*

PROOF. The proof is completely straightforward using the well-known fact that $f_n \to f$ uniformly on a set S if and only if for every $p \in {}^*S$ and every infinite ν, $f_\nu(p) \approx {}^*f(p)$ [**8**, Theorem 4.6.1].

We need one more definition before proving Egoroff's Theorem.

I.5 DEFINITION. Suppose A is a (possibly external) subset of ${}^*X$. Let $S(A)$ denote the set of all standard points in A. That is, $S(A) = \{x \in X \mid {}^*x \in A\}$. Note $S(A)$ is just the standard part of A with respect to the discrete topology on X.

I.6 PROOF OF EGOROFF'S THEOREM. Suppose $f_n \to f$ pointwise almost everywhere. Hence there is a set $A \in \mathfrak{F}$ such that $\mu(A) = 0$ and $f_n \to f$ pointwise on $X \setminus A$. Let E denote the set of points of intrinsic nonuniformity. Then $S(E) \subseteq A$. Thus $S(E)$ has measure zero and by II.3 E has S-measure zero, completing the proof by I.4.

II. The distribution of standard points in ${}^*X$. The purpose of this section is to study the relationship between the measure (in a sense to be defined below) of a set $E \subseteq {}^*X$ and the standard measure of $S(E)$. Intuitively, the standard points are evenly distributed in ${}^*X$ and one might, therefore, expect the measures of E and $S(E)$ to be infinitely close for a reasonable class of sets E.

II.1 DEFINITION. Let $\mathcal{A}$ be the (external) algebra, $\mathcal{A} = \{{}^*A \mid A \in \mathfrak{F}\}$, and let $\mathcal{S}$ be the (external) σ-algebra generated by $\mathcal{A}$. Using the Loeb-Carathéodory extension process there is an (external) real-valued σ-additive measure ${}^0({}^*\mu)\colon \mathcal{S} \to [0, \infty)$ [**5**], see also [**8**, §5.1], called S-measure. Notice ${}^0({}^*\mu)(A) = 0$ if and only if A has S-measure zero in the sense of I.3.

II.2 THEOREM. *Suppose* $E \in \mathcal{S}$. *Then* $S(E) \in \mathfrak{F}$ *and* ${}^0({}^*\mu)(E) = \mu(S(E))$.

PROOF. First, let $\mathfrak{T}_1 = \{E \in \mathcal{S} \mid S(E) \in \mathfrak{F}\}$. $\mathfrak{T}_1$ is a σ-algebra since $S({}^*X \setminus A) = X \setminus S(A)$, $S(A_1 \cap A_2) = S(A_1) \cap S(A_2)$ and $S(\bigcup_{n=1}^{\infty} A_n) = \bigcup_{n=1}^{\infty} S(A_n)$. Hence $\mathcal{S} = \mathfrak{T}_1$. Now we have two finite measures defined on $\mathcal{S}$, $\mu_1(E) = {}^0({}^*\mu)(E)$ and $\mu_2(E) = \mu(S(E))$. By the uniqueness part of the Caratheodory Extension Theorem, we have $\mu_1 = \mu_2$ completing the proof. Notice the importance here that $\mu(X)$ is finite.

II.3 EXAMPLE. Let E be as in I.6, the set of points of intrinsic nonuniformity for f and (f_n) where $f_n \to f$ almost everywhere. Then $E \in \mathcal{S}$ and ${}^0({}^*\mu)(E) = \mu(S(E)) = 0$.

PROOF. Let $A_{n,k} = \{x \in X | \exists r \geqslant k | f_r(x) - {}^*f(x)| \geqslant 1/n\}$. Claim:

$$E = \bigcup_{n=1}^{\infty} \bigcap_{k=1}^{\infty} {}^*A_{n,k}.$$

PROOF OF CLAIM. If $x \in E$ then there is an infinite ρ such that $f_\rho(x) \approx {}^*f(x)$. Therefore, there is a finite n such that $|f_\rho(x) - {}^*f(x)| \geqslant 1/n$. Thus $x \in A_{n,\rho} \subseteq {}^*A_{n,k}$ for every finite k. $\therefore x \in \bigcap_{k=1}^{\infty} {}^*A_{n,k}$. Conversely, suppose

$$x \in \bigcup_{n=1}^{\infty} \bigcap_{k=1}^{\infty} {}^*A_{n,k}.$$

Therefore there is an n such that $x \in \bigcap_{k=1}^{\infty} {}^*A_{n,k}$. Let $T = \{k | \, |f_k(x) - {}^*f(x)| \geqslant 1/n\}$. T is internal and contains arbitrarily large finite positive integers k. Therefore T contains some infinite positive integer ν and $|f_\nu(x) - {}^*f(x)| \geqslant 1/n$. So $x \in E$.

One of the difficulties in applying the techiques of Nonstantard Analysis to standard problems is converting a nonstandard object into a standard one. In particular, if F: ${}^*X \to \mathbf{R}$ is Loeb measurable and we define f: $X \to \mathbf{R}$ by $f(x) = F(x)$ then we have very little control over f. In fact, f need not be measurable and ${}^*f$ need have little relationship to F. One consequence of Theorem II.2 is that the situation is much better if F is $\mathcal{S}$-measurable. More precisely, we have

II.4 THEOREM. *Suppose F: ${}^*X \to \mathbf{R}$ is $\mathcal{S}$-measurable and f: $X \to \mathbf{R}$ is defined by $f(x) = F(x)$. Then*

(1) *f is $\mathcal{F}$-measurable,*

(2) *$\{x \in {}^*X | {}^*f(x) \approx F(x)\}$ has S-measure zero.*

PROOF. (1) Let $t \in \mathbf{R}$ and $A = \{x | f(x) > t\}$. Notice $A = S(E)$ where $E = \{x | F(x) > t\} \in \mathcal{S}$ by assumption. Hence, by Theorem II.2 $A \in \mathcal{F}$ and f is $\mathcal{F}$-measurable.

(2) Let ${}^0({}^*f)$: ${}^*X \to \mathbf{R} \cup \{\infty\}$ be given by

$${}^0({}^*f)(x) = \begin{cases} \mathrm{St}({}^*f(x)) & \text{if } {}^*f(x) \text{ is finite,} \\ \infty & \text{if } {}^*f(x) \text{ is infinite.} \end{cases}$$

A straightforward argument shows ${}^0({}^*f)$ is $\mathcal{S}$-measurable. Let $E = \{x | {}^0({}^*f)(x) \neq F(x)\}$. Then $E \in \mathcal{S}$ and $S(E) = \varnothing$. So, E has S-measure zero by Theorem II.2. But $E = \{x | {}^*f(x) \approx F(x)\}$ completing the proof.

The obvious question to ask is whether Theorem II.2 can be extended to a larger class of sets. A natural such question is whether for internal *measurable sets E, ${}^*\mu(E) \approx \mu(S(E))$. Unfortunately, the possible results in this direction are sharply circumscribed by the following examples.

II.5 EXAMPLE. Suppose $X = [0, 1]$ and ${}^*X$ is an enlargement of X. Then for every $B \subseteq [0, 1]$ and $t \in {}^*[0, 1]$ there is an internal, *Borel set E such that $S(E) = B$ and ${}^*\mu(E) = t$.

PROOF. A straightforward enlargement argument produces *finite sets F_1, F_2 such that $S(F_1) = B$, $S(F_2) = [0, 1] \setminus B$ and $F_1 \cap F_2 = \varnothing$. Let

$$E = ({}^*[0, t] \cup F_1) \setminus F_2.$$

II.6 EXAMPLE. Suppose $X = \{0, 1\}^{(\infty)}$ (i.e. an element $x \in X$ is a sequence $x = (x_1, x_2, \ldots)$ of 0's and 1's). Let X have the obvious probability measure. Let ${}^*X$ be any proper nonstandard extension of X and let ν be any infinite positive integer. Let $E = \{x \in {}^*X \mid x_\nu = 1\}$. It is well known and easy to prove using the Kolmogorov Zero-One Law that $S(E)$ has inner measure zero and outer measure one. But ${}^*\mu(E) = 1/2$.

II.7 EXAMPLE. We construct a nonstandard model ${}^*\mathfrak{M} = \mathfrak{M}^J/D$, where J is countable, such that there is an internal, *open set $V \subseteq {}^*[0, 1]$ with $S(V) = \varnothing$ but ${}^*\mu(V) \approx 1$ (μ is Lebesgue measure). Let $X = [0, 1]$ and let J be the set of all finite unions of disjoint open intervals with rational endpoints. Thus a typical element U of J is a set $\bigcup_{n=1}^{k}(a_n, b_n)$ with $a_1 < b_1 < a_2 < \cdots < b_n$ all rational. Notice J is countable. If $x_1, x_2, \ldots, x_k \in [0, 1]$ let $\mathfrak{U}(x_1, x_2, \ldots, x_k) = \{U \in J \mid x_1, x_2, \ldots, x_k \notin U, \mu(U) > 1 - 1/k\}$. Clearly the sets $\mathfrak{U}(x_1, x_2, \ldots, x_k)$ are nonempty and have the finite intersection property. Let D be any ultrafilter containing all the sets $\mathfrak{U}(x_1, x_2, \ldots, x_k)$. Let ${}^*\mathfrak{M} = \mathfrak{M}^J/D$ and let $V \subseteq {}^*X$ be the *open set represented by the function $F: J \to P(X)$ given by $F(U) = U$. It is immediate from the construction of V that ${}^*\mu(V) \approx 1$ and $S(V) = \varnothing$.

Thus, in general, there is no relationship between ${}^*\mu(E)$ and $\mu(S(E))$. However, if ${}^*X$ is a minimal nonstandard model (defined below) we do have some positive results.

II.8 DEFINITION. Suppose $J = \{1, 2, 3, \ldots\}$ and D is an ultrafilter on J. D is said to be *minimal*, see **[1]**, **[9]**, **[10]**, provided whenever $f: J \to J$ there is a set $A \in D$ such that either f is constant on A or f is one-to-one on A. If either the Continuum Hypothesis or Martin's Axiom holds there are many minimal ultrafilters on J **[1]**, **[9]**, **[10]**. If D is a minimal ultrafilter on J, the nonstandard model $\mathfrak{M}^J/D$ is said to be *minimal*. We use below the fact that minimal ultrafilters are Ramsey **[2]** and therefore satisfy the strong Ramsey theorem proved by Mathias **[6]**, **[7]**.

II.9 PROPOSITION. *Suppose ${}^*\mathfrak{M}$ is a minimal nonstandard model and $E \subseteq {}^*X$ is an internal *measurable set such that ${}^*\mu(E) \approx 0$. Then $S(E)$ is a μ-null set.*

PROOF. Let $e = {}^*\mu(E)$, let $(E_1, E_2, \ldots)$ represent E and let $e_n = \mu(E_n)$. Hence, e is represented by $(e_1, e_2, \ldots)$. Define $f: J \to J$ by $f(n) =$ largest k such that $e_n < 1/2^k$. Since D is minimal there is a set $A \in D$ such that f is constant on A or f is one-to-one on A. Since $e \approx 0$ the first alternative is impossible. Hence f is one-to-one on A. Therefore

$$\sum_{n \in A} e_n < \sum_{k=1}^{\infty} 1/2^k = 1.$$

Now, suppose $\varepsilon > 0$ is standard. Then there is a set $B \in D$ such that

$$\sum_{n \in B} e_n < \varepsilon.$$

But $x \in S(E)$ implies $x \in \bigcup_{r \in B} E_n$. Thus $S(E) \subseteq \bigcup_{n \in B} E_n$ but $\mu(\bigcup_{n \in B} E_n) \leqslant \sum_{n \in B} e_n \leqslant \varepsilon$. This completes the proof.

If $^*\mathfrak{M}$ is a minimal nonstandard model, μ is a Borel measure on a compact metric space and E is Loeb measurable then considerably more can be said about the relationship between the measures of E and $S(E)$. The first step is the following lemma.

II.10 LEMMA. *Suppose $^*\mathfrak{M}$ is minimal nonstandard model, μ is a Borel measure on a compact metric space K and E is a Loeb measurable set such that $S(E) = \varnothing$. Then* ${}^0(^*\mu)(E) = 0$.

PROOF. Suppose E is Loeb measurable, $S(E) = \varnothing$ and ${}^0(^*\mu)(E) > 0$. Then there is a standard $\varepsilon > 0$ and an internal, *measurable set $F \subseteq E$ such that $^*\mu(F) > \varepsilon$. Let (F_n) be a sequence of Borel subsets of K which determines F as an element of $^*\mathfrak{M}$. We then have that the set $Y = \{n \mid \mu(F_n) \geqslant \varepsilon\}$ is an element of the minimal ultrafilter D which is used to construct $^*\mathfrak{M}$. The Ramsey Theorem for D due to Mathias [6] will be used to show that there exists $Z \in D$ with $Z \subseteq Y$ and

$$\bigcap \{F_n \mid n \in Z\} \neq \varnothing.$$

Since this intersection is contained in $S(F)$, this contradicts our assumption that $S(F) = \varnothing$.

Given an infinite set $W \subseteq \mathbf{N}$, set $[W]^\omega = \{V \mid V \text{ is an infinite subset of } W\}$. On $[N]^\omega$ put the usual topology: the basic open neighborhoods of $W \in [N]^\omega$ are the sets

$$\{V \in [N]^\omega \forall k \leqslant n\, (k \in V \leftrightarrow k \in W)\}$$

for $n = 1, 2, \ldots$. The Ramsey Theorem of Mathias implies that if $\mathfrak{R} \subseteq [N]^\omega$ is analytic relative to this topology, there exists some $W \in D$ such that either $[W]^\omega \subseteq \mathfrak{R}$ or $[W]^\omega \cap \mathfrak{R} = \varnothing$. (By [2] a minimal ultrafilter is Ramsey; for a proof of Mathias' result that Ramsey ultrafilters have this much stronger property see [7].)

For our purposes we use the family

$$\mathfrak{R} = \{W \in [N]^\omega \mid \bigcap \{F_n \mid n \in W\} \neq \varnothing\}.$$

First we show that $\mathfrak{R}$ is analytic. Consider the set $\mathfrak{S} \subseteq [N]^\omega \times K$ defined by

$$\mathfrak{S} = \{(W, x) \mid x \in \bigcap \{F_n \mid n \in W\}\}.$$

Since each F_n is a Borel subset of K, $\mathfrak{S}$ is a Borel set in the product space $[N]^\omega \times K$. Also $\mathfrak{R}$ is the image of $\mathfrak{S}$ under the coordinate projection from $[N]^\omega \times K$ onto $[N]^\omega$. Since K is a compact metric space, it follows that $\mathfrak{R}$ is analytic in $[N]^\omega$ [4, Chapter XIII].

Now apply Mathias' theorem, obtaining a set $W \in D$ such that $[W]^\omega \subseteq \mathfrak{R}$ or $[W]^\omega \cap \mathfrak{R} = \varnothing$. In the first case we have $Z = Y \cap W \in D$ and $\bigcap \{F_n \mid n \in Z\} \neq \varnothing$ as desired. Thus it suffices to prove the second case is impossible. For any $W \in D$ the set $Z = Y \cap W$ is infinite and $\mu(F_n) \geqslant \varepsilon > 0$ holds for every $n \in Z$. Since μ is a finite measure it follows that

$$\mu\Big(\bigcap_{n \in N} \bigcup \{F_k \mid k \in Z \text{ and } k \geqslant n\}\Big) \geqslant \varepsilon.$$

Thus we may choose x and an infinite subset V of Z with $x \in F_k$ for all $k \in V$. Therefore $V \in [W]^\omega \cap \mathfrak{R}$, which completes the proof.

II.11 REMARK. In case μ is a regular Borel measure on K, a relatively simple case of Mathias' theorem can be used in the proof of II.10. In that case we may assume that the sets F_n are closed (replacing ε by $\varepsilon/2$ and each F_n by a closed subset). Then the family $\mathfrak{R}$ is actually a closed subset of $[N]^\omega$ as can be seen by a direct proof.

II.12 THEOREM. *Suppose* $^*\mathfrak{M}$ *is a minimal nonstandard model and* μ *is a Borel measure on a compact metric space* K. *If* $E \subseteq {}^*K$ *is Loeb measurable then*

$$\underline{\mu}(S(E)) \leqslant {}^0({}^*\mu(E)) \leqslant \bar{\mu}(S(E))$$

where $\underline{\mu}$, $\bar{\mu}$ *are the inner and outer measures for* μ.

PROOF. It suffices to prove $\underline{\mu}(S(E)) \leqslant {}^0({}^*\mu(E))$ since the other inequality follows from this one applied to $^*K \setminus E$.

Let B be a standard measurable set such that $B \subseteq S(E)$ and $\mu(B) = \underline{\mu}(S(E))$. Let $A = {}^*B \setminus E$. Notice $S(A) = \varnothing$ and A is Loeb measurable so by Lemma II.10 ${}^0({}^*\mu)(A) = 0$. Hence since ${}^0({}^*\mu)(A) + {}^0({}^*\mu)(E) \geqslant {}^0({}^*\mu)(B) = \underline{\mu}(S(E))$ we have ${}^0({}^*\mu)(E) \geqslant \underline{\mu}(S(E))$ completing the proof.

Example II.6 shows that even if E is internal and *measurable, $S(E)$ need not be μ-measurable. The following Corollary gives a necessary and sufficient condition for $S(E)$ to be μ-measurable when E is an internal, *Borel set in a minimal nonstandard model $^*\mathfrak{M}$.

II.13 COROLLARY. *Let* $^*\mathfrak{M}$, μ *and* K *be as in* II.10. *For each internal* *Borel *set* $E \subseteq {}^*K$, $S(E)$ *is measurable with respect to the completion of* μ *if and only if there is a standard Borel set* B *such that* E *and* $^*B$ *differ by a set of infinitesimal* $^*\mu$ *measure.*

PROOF. If such a set B exists, then $S(E)$ equals B up to a μ-null set, by II.9.

For the converse, suppose $S(E)$ is measurable with respect to the completion of μ and let $B \subseteq S(E)$ be a Borel set such that $S(E) \setminus B$ is a μ null set. Then $S({}^*B \setminus E)$ is empty and $S(E \setminus {}^*B)$ is a μ-null set, so that the symmetric difference of $^*B$ and E has infinitesimal $^*\mu$ measure, by II.12.

II.14 REMARK. Some restriction on the measure space of μ is necessary in order that II.10 should be true. For example, take Borel subsets E_n of $\{0, 1\}^{(\infty)}$ as in Example II.6, so that the internal set E determined by (E_n) has internal measure $1/2$, yet $S(E)$ has inner measure 0 and outer measure 1. Then consider the measure space $\Omega = \{0, 1\}^{(\infty)} \setminus S(E)$ with the restricted measure μ. Let $E_n' = E_n \setminus S(E)$, so (E_n') are measurable subsets of Ω. If E' is the internal *measurable subset of $^*\Omega$ determined by the sequence (E_n'), then $S(E') = \varnothing$ yet $^*\mu(E') = 1/2$.

II.15 REMARK. Lemma II.10 which is the key step in the proof of Theorem II.12 has a very nice standard interpretation as follows. Suppose D is a minimal ultrafilter on the set $\{1, 2, 3, \ldots\}$ and $E_1, E_2, \ldots$ is a sequence of Borel subsets of a compact metric space K with μ a finite Borel measure on K. If $\inf \mu(E_n) > 0$ then there is a point $x \in K$ such that $\{n | x \in E_n\} \in D$. Notice this is a strengthening of the usual result that there is a point $x \in K$ which is in infinitely many E_n's.

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