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KNOTS WITH HEEGAARD GENUS 2 COMPLEMENTS ARE INVERTIBLE

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ABSTRACT. Let K be a polyhedral oriented knot in S^3 and $N(K)$ be a regular neighborhood of K. If $S^3 \sim \mathring{N}(K)$ can be constructed by attaching a single 2-handle to a genus two handlebody, then there is a homeomorphism of S^3 onto itself mapping K onto itself and reversing the orientation of K.

We prove the title. A somewhat more careful statement is the following.

THEOREM. *Let $K \subset S^3$ be a polyhedral knot and let $N(K)$ be a regular neighborhood of K. If $S^3 \sim \mathring{N}(K)$ can be constructed by attaching a single 2-handle to a genus two handlebody, then K is invertible.*

PROOF. By a meridian of the knot K we mean a polyhedral disk D in $N(K)$ with $\partial D \subset \partial N(K)$ and $N(K) \sim N(D)$ is homeomorphic with a ball. ($N(D)$ is a regular neighborhood of D in $N(K)$.) We show how to construct a self-homeomorphism of $S^3 \sim \mathring{N}(K)$ which maps the boundary of a meridian of the knot onto its inverse. It is easy to see that such a homeomorphism can be extended to an involution of S^3 taking the (oriented) knot to its inverse. Let H_2 be a genus two solid handlebody, let D_1 and D_2 be meridian disks for H_2 and let γ be the simple closed curve on H_2 to which a 2-handle B is attached to get $S^3 \sim \mathring{N}(K)$. Let m be the boundary of a meridian of K. Without loss of generality we may assume that $m \subset \partial H_2 \sim \gamma$. Let $h\colon H_2 \to H_2$ be a rotation of H_2 through 180° about its axis. (Think of the standard picture of H_2. The axis passes through both holes of the handlebody.) Now h induces the symmetry η as defined in **[O & S**, p. 248]. Thus h can be assumed to map γ and m onto themselves while reversing their orientations (see also **[B & H**, §5]). The underlying reason for this is that the rotation inverts the Lickorish twists that generate the homeotopy group of the surface. Clearly then h may be extended to a homeomorphism $\tilde{h}$ which maps B onto itself while reversing orientation. This completes our proof.

Note. This result shows that knots such as 8_{20} and 10_{132} from the table of knots in **[Rolf]** are invertible. This does not appear evident from the presentations given. These knots are not torus knots or 2-bridge knots. The knot 8_{10} does not have a complement with Heegaard genus 2 but it is invertible. That it does not have such a complement follows from the fact that its second elementary ideal is proper **[Fox]**. Of course, the result above means that the noninvertible pretzel knots of Trotter

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[**Trot**] all have complements of Heegaard genus 3. It is certainly not easy to decide whether a given knot has a complement of Heegaard genus 2.

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