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A NOTE ON PERIODIC SOLUTIONS IN THE VICINITY OF A CENTER

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ABSTRACT. We show that a small polynomial perturbation of a center will result in periodic solutions provided the coefficients of the polynomial satisfy inequality (2) below. The proof is based on an application of degree theory.

1. Introduction. In her book [1] (see also [2]), Cronin stated the following. Let $h_1(x, y, \mu)$ and $h_2(x, y, \mu)$ be polynomials in x, y and μ and let $k_1(\sin\theta, \cos\theta, \mu)$, $k_2(\sin\theta, \cos\theta, \mu)$ be polynomials in $\sin\theta$, $\cos\theta$ and μ. Assume that $\eta(\mu)$ is a C^1 function and $\eta_0 = \eta(0) \neq 0$. Let $\omega = [1 + \mu\eta(\mu)]^{-1}$ and $\tau = \omega t$. Then the following two-dimensional nonlinear differential system

$$\begin{cases} x' = y + \mu[h_1(x, y, \mu) + k_1(\sin\tau, \cos\tau, \mu)], \\ y' = -x + \mu[h_2(x, y, \mu) + k_2(\sin\tau, \cos\tau, \mu)], \end{cases} \qquad \left(' = \frac{d}{dt}\right), \tag{1}$$

has, for all sufficiently small μ, at least one solution of period $2\pi(1 + \mu\eta(\mu))$.

For the proof of this result, Dr. Cronin used the results of degree theory, in particular, the homotopy theorem and the odd mapping theorem as applied to a function $N_\mu(C)$ which depends on the terms h_1, h_2, k_1 and k_2. The quantity C represents the initial data for the solution which is periodic precisely when $N_\mu(C) = 0$. But it seems that, in some cases, one cannot apply the odd mapping theorem. For example, if we let $\eta(\mu) \equiv 1$, $\omega = (1 + \mu)^{-1}$, $\tau = \omega\tau$ and consider

$$\begin{cases} x' = y + \mu(x + 3y + \cos\tau), \\ y' = -x + \mu(x - y - \sin\tau), \end{cases} \tag{$*$}$$

then $N_0(C) \equiv \binom{2\pi}{0}$. (From now on, we shall use the same notation as Cronin [1, pp. 91–93].) So the degree of $N_0(C)$ in any domain is 0. In this case, we cannot conclude from degree theory that the differential system ($*$) has, for all sufficiently small μ, at least one solution of period $2\pi(1 + \mu)$. The purpose of this paper is to compute directly $N_0(C)$. We will show that if the coefficients of h_1 and h_2 satisfy a simple inequality, then the differential system (1) has a periodic solution for μ sufficiently small. The precise statement of our result is given in the next section, but we note that the inequality required is a generic condition on h_1 and h_2, and consequently most systems satisfy the conclusion of Cronin's Theorem. The proof of our result appears in §§3 and 4.

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2. Statement of the theorem. For nonnegative integers n and k, we define

$$\alpha_{k,n} = \begin{cases} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \binom{2n+1-k}{n-j} & \text{if } 0 \leqslant k \leqslant n, \\ \sum_{j=0}^{2n+1-k} (-1)^{k+j-n} \binom{k}{n-j} \binom{2n+1-k}{j} & \text{if } n+1 \leqslant k \leqslant 2n+1. \end{cases}$$

Then, it is not hard to show that

$$\alpha_{k,n} = \frac{4^n}{\pi i^k} \int_0^{2\pi} \cos^{2n+1-k}\theta \cdot (-1)^k \sin^k \theta \cdot e^{i\theta}\, d\theta$$

$$= \begin{cases} \dfrac{4^n}{\pi} \cdot 2\pi \cdot \dfrac{1 \cdot 3 \cdot 5 \cdots (2n+1)}{2 \cdot 4 \cdot 6 \cdots (2n+2)}, & \text{if } k = 0, \\ \dfrac{4^n \cdot i^k}{\pi} \cdot 2\pi \cdot \dfrac{1 \cdot 3 \cdot 5 \cdots (2n+1-k) \cdot 1 \cdot 3 \cdot 5 \cdots (k-1)}{2 \cdot 4 \cdot 6 \cdots (2n+2)}, & \\ & \text{if } k \text{ is even and } 0 < k < 2n+1, \\ \dfrac{4^n \cdot i^{k+1}}{\pi} \cdot 2\pi \cdot \dfrac{1 \cdot 3 \cdot 5 \cdots (2n-k) \cdot 1 \cdot 3 \cdot 5 \cdots k}{2 \cdot 4 \cdot 6 \cdots (2n+2)}, & \\ & \text{if } k \text{ is odd and } 0 < k < 2n+1, \\ \dfrac{(-1)^{n+1} \cdot 4^n}{\pi} \cdot 2\pi \cdot \dfrac{1 \cdot 3 \cdot 5 \cdots (2n+1)}{2 \cdot 4 \cdot 6 \cdots (2n+2)}, & \text{if } k = 2n+1. \end{cases}$$

Let

$$P_n(x,y) = \sum_{k=0}^{2n+1} a_{k,n} x^{2n+1-k} y^k,$$

$$Q_n(x,y) = \sum_{k=0}^{2n+1} b_{k,n} x^{2n+1-k} y^k,$$

and let

$$A_n = \sum_{k=0}^{n} (-1)^k (a_{2k,n}\alpha_{2k,n} - b_{2k+1,n}\alpha_{2k+1,n}),$$

$$B_n = \sum_{k=0}^{n} (-1)^k (a_{2k+1,n}\alpha_{2k+1,n} + b_{2k,n}\alpha_{2k,n}).$$

Now we can state the following theorem.

THEOREM. *Assume that $h_1(x,y,\mu)$ and $h_2(x,y,\mu)$ are polynomials in x, y and μ. When $\mu = 0$, let*

$$h_1(x,y,0) = P(x,y) + \sum_{n=0}^{N} P_n(x,y),$$

$$h_2(x,y,0) = Q(x,y) + \sum_{n=0}^{N} Q_n(x,y),$$

where $P(x, y)$, $Q(x, y)$ are polynomials in x, y in which each term is of even degree. Define A_n, B_n as above, and let $k_1(\sin\theta, \cos\theta, \mu)$ and $k_2(\sin\theta, \cos\theta, \mu)$ be polynomials in $\sin\theta$, $\cos\theta$ and μ. Assume that $\eta(\mu)$ is a C^1 function and $\eta_0 = \eta(0)$. Let $\omega = [1 + \mu\eta(\mu)]^{-1}$ and $\tau = \omega t$. If

$$(a_{0,0} + b_{1,0})^2 + (2\eta_0 - a_{1,0} + b_{0,0})^2 + \sum_{n=1}^{N} (A_n^2 + B_n^2) \neq 0, \tag{2}$$

then the following two-dimensional nonlinear differential system

$$\begin{cases} x' = y + \mu\{h_1(x, y, \mu) + k_1(\sin\tau, \cos\tau, \mu)\}, \\ y' = -x + \mu\{h_2(x, y, \mu) + k_2(\sin\tau, \cos\tau, \mu)\}, \end{cases}$$

has, for all sufficiently small $|\mu|$, at least one periodic solution of period $2\pi(1 + \mu\eta(\mu))$.

COROLLARY. *If, in the above theorem, we let*

$$h_1(x, y, 0) = a_0 + a_1 x + a_2 y + H.O.T.,$$
$$h_2(x, y, 0) = b_0 + b_1 x + b_2 y + H.O.T.,$$

and assume $(a_1 + b_2)^2 + (2\eta_0 - a_2 + b_1)^2 \neq 0$, then the conclusion of the above theorem holds.

REMARK. We call attention to the fact that this theory applies in cases where $\eta_0 = 0$.

3. Computation of $N_0(C)$. Let

$$C = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

and let $z = c_1 - ic_2$. When $\mu = 0$, we let

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x(\theta, 0, C) \\ y(\theta, 0, C) \end{pmatrix} = \begin{pmatrix} c_1 \cos\theta + c_2 \sin\theta \\ -c_1 \sin\theta + c_2 \cos\theta \end{pmatrix} = \begin{bmatrix} \frac{1}{2}(e^{i\theta}z + e^{-i\theta}\bar{z}) \\ \frac{i}{2}(e^{i\theta}z - e^{-i\theta}\bar{z}) \end{bmatrix}.$$

In particular, when $z = 1$, one has $x = \cos\theta$, $y = -\sin\theta$.

In Cronin's notation, one has

$$N_0(C) = \begin{pmatrix} 2\pi\eta_0 c_2 + H_1(c_1, c_2) + K_1 \\ -2\pi\eta_0 c_1 + H_2(c_1, c_2) + K_2 \end{pmatrix}$$

where

$$\begin{pmatrix} H_1(c_1, c_2) \\ H_2(c_1, c_2) \end{pmatrix} = \int_0^{2\pi} \begin{pmatrix} h_1(x, y, 0)\cos\theta - h_2(x, y, 0)\sin\theta \\ h_1(x, y, 0)\sin\theta + h_2(x, y, 0)\cos\theta \end{pmatrix} d\theta$$
$$= \int_0^{2\pi} \begin{bmatrix} \mathrm{Re}\{[h_1(x, y, 0) + ih_2(x, y, 0)]e^{i\theta}\} \\ \mathrm{Im}\{[h_1(x, y, 0) + ih_2(x, y, 0)]e^{i\theta}\} \end{bmatrix} d\theta$$

and

$$\begin{pmatrix} K_1 \\ K_2 \end{pmatrix} = \int_0^{2\pi} \begin{pmatrix} k_1(\sin\theta, \cos\theta, 0)\cos\theta - k_2(\sin\theta, \cos\theta, 0)\sin\theta \\ k_1(\sin\theta, \cos\theta, 0)\sin\theta + k_2(\sin\theta, \cos\theta, 0)\cos\theta \end{pmatrix} d\theta$$

is a constant vector.

Since h_1, h_2 are polynomials in x and y, it suffices to compute $\int_0^{2\pi} h(x,y)e^{i\theta}\,d\theta$, where $h(x,y)$ is a polynomial in x and y.

Given two nonnegative integers m and n, and let $h(x,y) = x^m y^n$. If $m+n$ is even, then it is trivial that $\int_0^{2\pi} h(x,y)e^{i\theta}\,d\theta = 0$. So we can assume that $m+n$ is odd.

If $h(x,y) = x^{2n+1-k}y^k$, where $0 < k < n$, then

$$\begin{aligned}
\int_0^{2\pi} h(x,y)e^{i\theta}\,d\theta &= \int_0^{2\pi} x^{2n+1-k}y^k e^{i\theta}\,d\theta \\
&= \int_0^{2\pi} \frac{1}{2^{2n+1-k}}(e^{i\theta}z + e^{-i\theta}\bar{z})^{2n+1-k}\left(\frac{i}{2}\right)^k (e^{i\theta}z - e^{-i\theta}\bar{z})^k e^{i\theta}\,d\theta \\
&= \frac{i^k\cdot\pi}{4^n}(z\bar{z})^n\bar{z}\cdot\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\binom{2n+1-k}{n-j} \\
&= \frac{i^k\pi}{4^n}(z\bar{z})^n\bar{z}\cdot\alpha_{k,n}.
\end{aligned}$$

Similarly, if $h(x,y) = x^{2n+1-k}y^k$, where $n+1 < k < 2n+1$, then

$$\begin{aligned}
\int_0^{2\pi} h(x,y)e^{i\theta}\,d\theta &= \int_0^{2\pi} x^{2n+1-k}y^k e^{i\theta}\,d\theta \\
&= \frac{i^k\pi}{4^n}(z\bar{z})^n\bar{z}\cdot\sum_{j=0}^{2n+1-k}(-1)^{k+j-n}\binom{k}{n-j}\binom{2n+1-k}{j} \\
&= \frac{i^k\pi}{4^n}(z\bar{z})^n\bar{z}\cdot\alpha_{k,n}.
\end{aligned}$$

Now assume

$$h_1(x,y,0) = \sum_{k=0}^{2n+1} a_{k,n}x^{2n+1-k}y^k = P_n(x,y),$$

$$h_2(x,y,0) = \sum_{k=0}^{2n+1} b_{k,n}x^{2n+1-k}y^k = Q_n(x,y), \qquad (n > 0).$$

Then

$$\begin{aligned}
&\int_0^{2\pi}[h_1(x,y,0)+ih_2(x,y,0)]e^{i\theta}\,d\theta \\
&\quad=\sum_{k=0}^{2n+1}(a_{k,n}+ib_{k,n})\int_0^{2\pi}x^{2n+1-k}y^k e^{i\theta}\,d\theta \\
&\quad=\sum_{k=0}^{2n+1}(a_{k,n}+ib_{k,n})\cdot\frac{i^k\pi}{4^n}\cdot(z\bar z)^n\bar z\cdot\alpha_{k,n} \\
&\quad=\frac{\pi}{4^n}(z\bar z)^n\bar z\cdot\left[\sum_{k=0}^{n}(-1)^k a_{2k,n}\alpha_{2k,n}-\sum_{k=0}^{n}(-1)^k b_{2k+1,n}\alpha_{2k+1,n}\right] \\
&\qquad+i\frac{\pi}{4^n}(z\bar z)^n\bar z\cdot\left[\sum_{k=0}^{n}(-1)^k a_{2k+1,n}\alpha_{2k+1,n}+\sum_{k=0}^{n}(-1)^k b_{2k,n}\alpha_{2k,n}\right] \\
&\quad=\frac{\pi}{4^n}(z\bar z)^n\bar z A_n+i\frac{\pi}{4^n}(z\bar z)^n\bar z B_n \quad \text{(where } A_n,\ B_n \text{ are as defined in §2)} \\
&\quad=\frac{\pi}{4^n}(z\bar z)^n\bar z(A_n+iB_n) \\
&\quad=\frac{\pi}{4^n}(c_1^2+c_2^2)^n(c_1+ic_2)(A_n+iB_n) \\
&\quad=\frac{\pi}{4^n}(c_1^2+c_2^2)^n[(c_1A_n-c_2B_n)+i(c_1B_n+c_2A_n)].
\end{aligned}$$

Therefore, the real and imaginary parts of $\int_0^{2\pi}[P_n(x,y)+iQ_n(x,y)]e^{i\theta}\,d\theta$ are $(\pi/4^n)(c_1^2+c_2^2)^n(c_1A_n-c_2B_n)$ and $(\pi/4^n)(c_1^2+c_2^2)^n(c_1B_n+c_2A_n)$ respectively.

Finally, we let $h_1(x,y,0)=P(x,y)+\sum_{n=0}^N P_n(x,y)$, and $h_2(x,y,0)=Q(x,y)+\sum_{n=0}^N Q_n(x,y)$ where $P(x,y)$ and $Q(x,y)$ are polynomials in x,y in which each term is of even degree, and $P_n(x,y)$, $Q_n(x,y)$ are defined as above. Then

$$\int_0^{2\pi}[h_1(x,y,0)+ih_2(x,y,0)]e^{i\theta}\,d\theta=\sum_{n=0}^{N}\int_0^{2\pi}[P_n(x,y)+iQ_n(x,y)]e^{i\theta}\,d\theta.$$

(Note that

$$\begin{aligned}
\int_0^{2\pi}P(x,y)e^{i\theta}\,d\theta=0=\int_0^{2\pi}Q(x,y)e^{i\theta}\,d\theta \\
=\sum_{n=0}^{N}\frac{\pi}{4^n}(c_1^2+c_2^2)^n[(c_1A_n-c_2B_n)+i(c_1B_n+c_2A_n)].
\end{aligned}$$

So the real and imaginary parts of $\int_0^{2\pi}[h_1(x,y,0)+ih_2(x,y,0)]e^{i\theta}\,d\theta$ are $\sum_{n=0}^N(\pi/4^n)(c_1^2+c_2^2)^n(c_1A_n-c_2B_n)$ and $\sum_{n=0}^N(\pi/4^n)(c_1^2+c_2^2)^n(c_1B_n+c_2A_n)$ respectively. Therefore,

$$H_1(c_1,c_2)=\sum_{n=0}^{N}\frac{\pi}{4^n}(c_1^2+c_2^2)^n(c_1A_n-c_2B_n),$$

$$H_2(c_1,c_2)=\sum_{n=0}^{N}\frac{\pi}{4^n}(c_1^2+c_2^2)^n(c_1B_n+c_2A_n),$$

where A_n, B_n are as defined in §2. So we have got the explicit formulas for $H_1(c_1,c_2)$ and $H_2(c_1,c_2)$.

4. Proof of the theorem. Now we have obtained the formulas for $H_1(c_1, c_2)$ and $H_2(c_1, c_2)$. So

$$N_0(C) = \begin{pmatrix} 2\pi\eta_0 c_2 + H_1(c_1, c_2) + K_1 \\ -2\pi\eta_0 c_1 + H_2(c_1, c_2) + K_2 \end{pmatrix}$$

$$= \begin{bmatrix} 2\pi\eta_0 c_2 + \pi(c_1 A_0 - c_2 B_0) + \sum_{n=1}^{N} \frac{\pi}{4^n}(c_1^2 + c_2^2)^n (c_1 A_n - c_2 B_n) + K_1 \\ -2\pi\eta_0 c_1 + \pi(c_1 B_0 + c_2 A_0) + \sum_{n=1}^{N} \frac{\pi}{4^n}(c_1^2 + c_2^2)^n (c_1 B_n + c_2 A_n) + K_2 \end{bmatrix}$$

$$= \pi \begin{bmatrix} c_1(a_{0,0} + b_{1,0}) + c_2(2\eta_0 + b_{0,0} - a_{1,0}) + \sum_{n=1}^{N} \frac{(c_1^2 + c_2^2)^n}{4^n}(c_1 A_n - c_2 B_n) \\ c_1(-2\eta_0 - b_{0,0} + a_{1,0}) + c_2(a_{0,0} + b_{1,0}) + \sum_{n=1}^{N} \frac{(c_1^2 + c_2^2)^n}{4^n}(c_1 B_n + c_2 A_n) \end{bmatrix}$$

$$+ \begin{pmatrix} K_1 \\ K_2 \end{pmatrix}.$$

If for some $n \geqslant 1$, $A_n^2 + B_n^2 \neq 0$, then $N_0(C)$ is dominated at least by $(c_1^2 + c_2^2)^n$. For all sufficiently large $\|C\|$, one has $N_0(C) \neq 0$, and we can apply the odd mapping theorem. On the other hand, if $A_n = B_n = 0$ for all $1 \leqslant n \leqslant N$ and if $(a_{0,0} + b_{1,0})^2 + (2\eta_0 + b_{0,0} - a_{1,0})^2 \neq 0$, then

$$N_0(C) = \pi \begin{pmatrix} c_1(a_{0,0} + b_{1,0}) + c_2(2\eta_0 + b_{0,0} - a_{1,0}) \\ c_1(-2\eta_0 - b_{0,0} + a_{1,0}) + c_2(a_{0,0} + b_{1,0}) \end{pmatrix} + \begin{pmatrix} K_1 \\ K_2 \end{pmatrix} = 0$$

for exactly one C. So, again, for all sufficiently large $\|C\|$, the function $N_0(C)$ can never be zero and the odd mapping theorem applies too. But if $a_{0,0} + b_{1,0} = 0 = 2\eta_0 + b_{0,0} - a_{1,0}$ and $A_n = B_n = 0$ for $1 \leqslant n \leqslant N$, then the function $N_0(C)$ is constantly equal to $\binom{K_1}{K_2}$. So its degree is either undefined or equal to zero. Hence no conclusion can be drawn in this case. This completes the proof.

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