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COMPACTIFICATIONS OF SYMMETRIZABLE SPACES

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ABSTRACT. In response to questions of Arhangel'skiĭ, we present examples of (1) (MA + $\neg$CH) a symmetrizable space which is not metrizable but has a completely normal compactification and (2) (CH) a symmetrizable space which is not metrizable but has a perfectly normal compactification. In the construction of (2), a technique is developed which can be used to obtain first countable compactifications of many interesting examples.

1. Introduction. A topological space X is called symmetrizable provided that there is a function $d: X \times X \to [0, \infty)$ such that for $(x, y) \in X \times X$, $d(x, y) = d(y, x)$, $d(x, y) = 0$ iff $x = y$, and $A \subseteq X$ is closed iff $d(x, A) > 0$ for each $x \in X - A$. All Moore spaces and all semimetrizable spaces are symmetrizable, and all T_2 first countable symmetrizable spaces are semimetrizable.

In [A], the following question is raised:

Question 1.1. Is every symmetrizable subspace of a completely normal compact space metrizable?

While this is an interesting question, we believe (and have been informed by a reliable source) that the question which was intended by Arhangel'skiĭ is the following:

Question 1.2. Is every symmetrizable subspace of a perfectly normal compact space metrizable?

In this note we give consistent answers to both of these questions by exhibiting an example (under MA + $\neg$CH) which answers Question 1.1 in the negative and an example (under CH) which answers Questions 1.2 and 1.1 in the negative. This would lead one to believe that there must be a ZFC example answering Question 1.1 in the negative, but no such example is known at this writing.

Of independent interest is an intermediate construction which is used in building the second example. We are able to obtain from this construction, first countable compactifications of many interesting spaces such as the "bow-tie" space and the "tangent disk" space.

2. The first example. We begin with a preliminary lemma.

LEMMA 2.1. *If X is a locally compact, completely normal space, then X^*, the one point compactification of X, is completely normal.*

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PROOF. Suppose $A \subseteq X^*$ and F, H are disjoint closed (in A) subsets of A. If $F \cup H \subseteq X$, the normality of $A \cap X$ will suffice to separate F and H. Suppose the ideal point $\infty \in H$. Then ∞ is not a limit point of F in X^*. Choose U_1, V_1 disjoint open in X^* with $F \subseteq U_1$, $\infty \in V_1$. Now choose U_2, V_2 disjoint open in $A \cap X$ with $F \subseteq U_2$, $H - \{\infty\} \subseteq V_2$. Now $U = U_1 \cap U_2$ and $V = V_1 \cup V_2$ separate F and H in A.

Since a normal Moore space is completely normal, we easily get a counterexample for Question 1.1.

EXAMPLE 2.2 (MA + $\neg$CH). Let X be a locally compact normal nonmetrizable Moore space. For instance, X could be an ω_1-Cantor tree. By Lemma 2.1, X^* is completely normal, and X is a symmetrizable, nonmetrizable subspace.

Any example which supplies a negative answer to Question 1.1 by looking at a one-point compactification must be of this type since a locally compact T_2 symmetrizable space is a Moore space **[A]**. In fact, if X is a symmetrizable nonmetrizable subspace of a completely normal compactum Z, and X is Čech-complete, a p-space, or a $w\Delta$-space, then X is a normal nonmetrizable Moore space **[Bu]**.

3. The second example. To answer Question 1.2, we seek a nonmetrizable symmetrizable space X and a compact space Z with $X \subseteq Z$ and Z perfectly normal. Obviously, we may assume that X is dense in Z since $\mathrm{Cl}_Z X$ will also do the job.

We observe that X and Z must both be first countable and hereditarily Lindelöf. First countability follows since singletons in Z are G_δ-sets in the compact space Z, and it is well known that perfect Lindelöf spaces are hereditarily Lindelöf. If X has a countable network, then X would have a countable base **[AH]**. Hence, we must have X a semimetrizable, hereditarily Lindelöf space which does not have a countable network. At this time there are three such examples in the literature, **[B]**, **[V]**, **[M]**, and all of them use CH. Hence, in our construction we will assume CH. The reader will note that in the initial stages of our construction we mimic Michael's technique **[M]**.

We begin with H, the Heath bow-tie space **[H]**. The points of H are the points of the plane. For a point $x = (x_1, x_2)$, we let $B(x, n)$ be the set given by

$$B(x, n) = \left\{ y \in \mathbf{R} \times \mathbf{R}: \|x - y\| < \frac{1}{n} \text{ and } \left| \frac{y_2 - x_2}{y_1 - x_1} \right| < \tan \frac{1}{2n} \right\}.$$

Geometrically, $B(x, n)$ is the point $\{x\}$ together with the interiors of two sectors of the circle centered at x with radius $1/n$ which have central angle $1/n$ and are bisected by a horizontal line, i.e., $B(x, n)$ looks like a bow-tie with $\{x\}$ at the knot.

It is easy to see that the topology generated by using $\{B(x, n): n \in N, x \in H\}$ is a semimetrizable topology on H. We will construct our example X as a subspace of H. It will be useful to consider bow-ties which have been rotated. To this end, we define a bow-tie B to have orientation α, if B can be obtained from $B(x, n)$, for some $x \in H$, $n \in N$, by a counterclockwise rotation through the angle α. In this case, we write $B = B(x, n, \alpha)$. In particular, $B(x, n) = B(x, n, 0)$ for each $x \in H$, $n \in N$. We will be considering only the case when $\alpha \in [0, \pi)$ since a bow-tie with any other orientation equals a bow-tie with orientation in this range.

We say a set $P \subseteq H$ is a p-neighborhood of a set $A \subseteq H$ provided that for each $x \in A$, there exists $n \in N$ and $\theta \in [0, \pi)$ with $B(x, n, \theta) \subseteq P$. It is clear that there is a collection $\{U_\alpha\colon \alpha < c\}$ of p-neighborhoods of $\mathbf{Q} \times \mathbf{Q}$ such that if P is a p-neighborhood of $\mathbf{Q} \times \mathbf{Q}$ then for some $\alpha < c$, $U_\alpha \subseteq P$. Moreover, if P is a p-neighborhood of $\mathbf{Q} \times \mathbf{Q}$, then the Euclidean interior of P is dense in $\mathbf{R} \times \mathbf{R}$.

We are now ready to construct the space X.

EXAMPLE 3.1 (CH). Use CH to index the "base" for the p-neighborhoods of $\mathbf{Q} \times \mathbf{Q}$, which was mentioned earlier, as $\{U_\alpha\colon \alpha < \omega_1\}$. By the fact that $\mathbf{R} \times \mathbf{R}$ is a complete metric space, we have that for any $\beta < \omega_1$ the set $\bigcap_{\alpha<\beta} U_\alpha$ is of second category in $\mathbf{R} \times \mathbf{R}$. For each $\beta < \omega_1$, choose $x_\beta \in (\bigcap_{\alpha<\beta} U_\alpha) - ((\mathbf{Q} \times \mathbf{Q}) \cup \{x_\alpha\colon \alpha < \beta\})$. The space $X = (\mathbf{Q} \times \mathbf{Q}) \cup \{x_\beta\colon < \omega_1\}$ viewed as a subspace of H. It can be seen from the arguments given in [M, Lemma 3.5] that X with the topology from any fixed orientation is a hereditarily Lindelöf, semimetrizable space which has no countable network.

We now exhibit a technique for building first countable compactifications on which we will elaborate more in the general situation in the next section. For now we apply the technique to H.

EXAMPLE 3.2. There is a first countable compactification Y of the space H. We shrink H onto $(0, 1) \times (0, 1) \subseteq [0, 1] \times [0, 1]$ and view X as a subset of $(0, 1) \times (0, 1)$. The set for Y is $[0, 1] \times [0, 1] \times S^1$ where S^1 is a circle which we denote by $S^1 = [0, 2\pi)$. For a point $(x_1, x_2, \theta) \in Y$ we define $U_n((x_1, x_2), \theta)$ by the following:

$$\begin{aligned} U_n((x_1, x_2), \theta) = \big[B((x_1, x_2), n, \theta/2) &- \{(x_1, x_2)\} \times S^1 \\ &\cup \{(x_1, x_2, \alpha)\colon |\alpha/2 - \theta/2| < 1/2n \\ &\qquad \text{or } |(\alpha - 2\pi)/2 - \theta/2| < 1/2n\} \big] \cap Y. \end{aligned}$$

Now $\{U_n((x_1, x_2), \theta)\colon n \in N, (x_1, x_2, \theta) \in Y\}$ is a base for a first countable topology on Y. Moreover $H \times \{0\}$ is homeomorphic to H and is dense in Y. It is easy to see that Y is a T_2 space. To see that Y is compact, suppose $\mathcal{U}$ is an open cover of Y. For each $x = (x_1, x_2) \in [0, 1] \times [0, 1]$, choose a finite subcollection $\mathcal{U}_x$ of $\mathcal{U}$ which covers $\{x\} \times S^1$. Now $\bigcup \mathcal{U}_x$ contains a solid torus. Moreover, since $[0, 1] \times [0, 1]$ is compact in its usual topology, finitely many of $\{\bigcup \mathcal{U}_x\colon x \in [0, 1] \times [0, 1]\}$ must cover Y. This gives rise to a finite subcover from $\mathcal{U}$, and the proof is complete.

At this point, we can supply the compactification Z of X which answers Question 1.2 in the negative.

EXAMPLE 3.3 (CH). There is a perfect compactification Z of X. The space Z is obtained from Y by identifying all points (x_1, x_2, θ) with $(x_1, x_2, 0)$ for $(x_1, x_2) \in [0, 1] \times [0, 1] - X$. (Recall that we have H and thus X contained in $[0, 1] \times [0, 1]$.) Call the quotient mapping which does the identification $\phi\colon Y \to Z$.

Since ϕ is a continuous, in fact perfect, mapping, Z is compact. It is clear that $X \times \{0\}$ is dense in Z. Thus Z is compactification of X.

We now show that Z is hereditarily Lindelöf. To do this, we will show that if $\mathcal{U}$ is a collection of basic open sets in Z, then there is a countable $\mathcal{U}_1 \subseteq \mathcal{U}$ with $\bigcup \mathcal{U}_1 = \bigcup \mathcal{U}$. If $x \in [0, 1] \times [0, 1] - X$, then a base at $(x, 0)$ in Z is given by $\{(B \times \{0\}) \cup ((B \cap X) \times S^1)$: B is a Euclidean ball about $x\}$. Thus it is sufficient to consider collections of basic open sets centered at points of $X \times S^1$. Suppose $\mathcal{U} = \{\phi U_{n_\alpha}(x_\alpha, \theta_\alpha)$: $\alpha \in A\}$. For each α, let $S_\alpha = \phi U_{n_\alpha}(x_\alpha, \theta_\alpha) \cap (\{x_\alpha\} \times S^1)$. Let $\mathcal{W} = \{\phi U_{n_\alpha}(x_\alpha, \theta_\alpha) - S_\alpha$: $\alpha \in A\}$. Note that for $W \in \mathcal{W}$, $W = (V \times \{0\}) \cup ((V \cap X) \times S^1)$ for some Euclidean open $V \subseteq [0, 1] \times [0, 1]$. Hence there is a countable $\mathcal{W}_1 \subseteq \mathcal{W}$ with $\bigcup \mathcal{W}_1 = \bigcup \mathcal{W}$. Now if $z \in \bigcup \mathcal{U} - \bigcup \mathcal{W}_1$, then $z \in S_\alpha$ for some $\alpha \in A$. Since S^1 is hereditarily Lindelöf, we will be finished when we have shown that there exist only countably many $x_\alpha \in X$ such that $S_\alpha - \bigcup \mathcal{W}_1 \neq \varnothing$. Suppose $A_1 \subseteq A$ with A_1 uncountable and $S_\alpha - \bigcup \mathcal{W}_1 \neq \varnothing$ for every $\alpha \in A_1$. For each $\alpha \in A_1$, $S_\alpha = \{x_\alpha\} \times I_\alpha$ for some open interval $I_\alpha \subseteq S^1$. Since S^1 is separable, there exists $\theta \in S^1$ such that for an uncountable $A_2 \subseteq A_1$, $\theta \in I_\alpha$ for every $\alpha \in A_2$. Now $\{(x_\alpha, \theta)$: $\alpha \in A_2\}$ is an uncountable relatively discrete subset of $X \times \{\theta\}$, but $X \times \{\theta\}$ is hereditarily Lindelöf. This contradiction establishes the result.

REMARK. The reader will note that a large portion of the proof that Z is hereditarily Lindelöf is really the proof that Z is hereditarily $\aleph_1$-compact. If this is done first, we can use the fact that Z is an $\mathcal{F}$-space [HS] to get the result since it is shown in [HS] that hereditarily $\aleph_1$-compact $\mathcal{F}$-spaces are hereditarily Lindelöf.

A somewhat stronger statement can be made concerning the weak base properties of Z. This is actually an $\mathcal{F}_r$-space [D]. We point this out since $\mathcal{F}_r$-space is one of the strongest weak base assumptions which does not imply metrizability in the presence of compactness. (Symmetrizable, of course, does imply metrizable in the presence of compactness.)

4. Questions and applications of Y.

The construction used in Y is a compactification technique which uses the idea that the top and bottom of the lexicographic square compactifies the Sorgenfrey line by taking advantage of the orientation of the basic neighborhoods. This idea can be used with many of the interesting examples currently in the literature for which "orientation" makes sense and the union of neighborhoods of the assorted orientations yields a neighborhood in some compact space. In the example in this paper, we saw that the bow-ties of the various orientations when unioned at a particular point yielded a Euclidean ball.

We see from the above that many popular examples have first countable compactifications. In contrast to this, it is of interest to point out an example given in [vDP] of a Lindelöf semimetrizable space, with a countable network, for which all compactifications contain βN. Hence, not only the character, but also the tightness will be large.

We conclude with a couple of questions.

Question 4.1. Can Questions 1.1 and 1.2 be answered in ZFC?

Question 4.2. Can MA + ¬CH be used to obtain a positive answer to Question 1.2?

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