PETTIS INTEGRATION[1]

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ABSTRACT. The functions that are Pettis integrable with respect to perfect measures are characterized.

Since its introduction by B. J. Pettis in 1938, the Pettis integral of a weakly measurable function has proved remarkably resistant to analysis. Nearly forty years passed without a significant improvement on Pettis's original work. Recently, the Pettis integral has begun to come into its own. Charles Stegall has proved that on perfect measure spaces the range of the indefinite Pettis integral is relatively compact [3, Proposition 35]. In this paper we shall carry on this work. With the help of fundamental theorems due to Fremlin and James, we completely characterize Pettis integrability in terms of convergence of simple functions. Our work shows that a bounded function f from a finite perfect measure space to a Banach space X is Pettis integrable if and only if there is a bounded sequence (f_n) of simple functions such that $\lim_{n\to\infty} x^*f_n = x^*f$ a.e. for each x^* in X^*. This parallels the situation for Bochner integrability. A bounded function f from a finite measure space into a Banach space X is Bochner integrable if and only if there is a null set A and a sequence (f_n) of simple functions such that $\lim_{n\to\infty} x^*f_n(\omega) = x^*f(\omega)$ for all x^* in X^* and for all ω not in A.

I. Terminology. Let (Ω, Σ, μ) be a finite measure space and let X be a Banach space. A function $f(\cdot)$ from Ω into X is *weakly measurable* if the scalar function $x^*f(\cdot)$ is measurable for each x^* in the dual space X^*. The function f is *Pettis integrable* if for each E in Σ there is an element of X, denoted $\int_E f\, d\mu$, that satisfies $\langle x^*, \int_E f\, d\mu \rangle = \int_E x^*f\, d\mu$ for every x^* in X^*.

Another form of measurability can be attributed to some vector-valued functions. If f is almost everywhere the limit (in the norm topology of X) of a sequence of simple functions, then f is *strongly measurable*. The Pettis Measurability Theorem [5, Theorem 1.1] says that a function is strongly measurable if and only if it is weakly measurable and off a null set it has separable range. If f is strongly measurable and if $\int \|f\|\, d\mu < \infty$ then the *Bochner integral* of f exists as an element of X [1, Theorem 2, p. 45], and f is trivially Pettis integrable. The Bochner integral has received considerable study—see [1].

A family Φ of scalar functions is *uniformly integrable* if $\lim_{\mu(E)\to 0}\int_E|\varphi|\,d\mu = 0$ uniformly for φ in Φ. Note that pointwise uniformly bounded families are necessarily uniformly integrable. An easy consequence of the Orlicz-Pettis theorem is that if $f\colon \Omega \to X$ is Pettis integrable then $\{x^*f\colon x^* \in \text{Ball}(X^*)\}$ is uniformly integrable [1, Theorem 8, p. 55].

All of our work builds on a result of D. M. Fremlin, and for this we need the concept of a perfect measure space. A finite measure space (Ω, Σ, μ) is *perfect* if for each measurable $\psi\colon \Omega \to R$ and for each set $E \subset R$ such that $\psi^{-1}(E) \in \Sigma$, there is a Borel set $B \subset E$ such that $\mu[\psi^{-1}(B)] = \mu[\psi^{-1}(E)]$. The class of perfect measure spaces is very broad; in particular, all Radon measure spaces are perfect [7, Theorem 10]. Fremlin's theorem [2, Theorem 2F] says that every sequence of measurable scalar functions on a finite perfect measure space has either a subsequence with no measurable pointwise cluster point, or a subsequence that converges almost everywhere.

The following lemma adapts this theorem to our needs. Note that if $f\colon \Omega \to X$ and if (x_k^*) is any bounded sequence in X^*, then it follows from Alaoglu's theorem that every subsequence of (x_k^*f) has a pointwise cluster point. Combining this with Fremlin's theorem, we have

LEMMA 1. *Let (Ω, Σ, μ) be a finite perfect measure space, and let $f\colon \Omega \to X$ be weakly measurable. If (x_k^*) is a bounded sequence in X^*, then there is a subsequence $(x_{k_j}^*)$ such that $\lim_{j\to\infty}x_{k_j}^*f$ exists a.e. If x_0^* is any weak*-cluster point of $(x_{k_j}^*)$, then $\lim_{j\to\infty} x_{k_j}^*f = x_0^*f$ a.e.*

II. Convergence theorems. A bounded vector-valued function is Bochner integrable if and only if it is almost everywhere the norm limit of a sequence of simple functions. This statement concerns only the measurability of the function in question–all strongly measurable bounded functions are Bochner integrable. The state of affairs for Pettis integrability is more complex. R. S. Phillips gave an example in 1940 of a bounded weakly measurable function that is not Pettis integrable [6, Example 10.8]. Phillips's example is so pertinent that we reproduce it here.

EXAMPLE 2. *A bounded, nonintegrable function.*

Sierpinski constructed [9, pp. 9–10] a subset B of $[0, 1] \times [0, 1]$ with the properties

(1) for each number t_0 in $[0, 1]$ the set $\{s\colon (s, t_0) \in B\}$ is countable and

(2) for each number s_0 in $[0, 1]$ the set $\{t\colon (s_0, t) \notin B\}$ is countable.

Define $f\colon [0, 1] \to l_\infty[0, 1]$ by

$$[f(s)](t) = \chi_B(s, t).$$

We show first that f is weakly measurable. Phillips proved [7] that every element β of the space $ba[0, 1]$ $(= l_\infty^*[0, 1])$ may be uniquely written as the sum of two measures, $\beta = \beta_1 + \beta_2$, such that β_1 has countable support and β_2 vanishes on null sets. Since the set $\{t\colon (s_0, t) \notin B\}$ is countable for each number s_0, we have

$$\int f(s_0)\,d\beta_2(t) = \int 1\,d\beta_2(t).$$

The countability of the support of β_1, together with condition (1), implies that

$$\int f(s_0)\, d\beta_1(t) = 0$$

for almost all numbers s_0. It follows that

$$\int f(s_0)\, d\beta(t) = \int 1\, d\beta_2(t)$$

for almost all s_0 and that f is weakly measurable.

To see that f is not Pettis integrable, for each t_0 in $[0, 1]$ let e_{t_0} be the "evaluation at t_0" functional on $l_\infty[0, 1]$. Then from condition (2) we have

$$\int e_{t_0} \circ f(s)\, ds = \int \chi_B(s, t_0)\, ds = 0.$$

In other words, if the Pettis integral of f exists, it must be the 0-element of $[0, 1]$. However, an easy calculation shows that $0 \notin \overline{co}\, f([0, 1])$. There thus exists an element x^* of $l_\infty^*[0, 1]$ such that $x^*f(s) > 0$ for all s in $[0, 1]$, and $\int x^*f(s)\, ds > 0$. This contradiction shows that f cannot be Pettis integrable.

The next theorem is the keystone of this paper, a Pettis analogue of Vitali's classic convergence theorem. Note that conditions (a) and (b) of this theorem guarantee that for each x^* the sequence of scalar integrals $(\int_E x^*f_n\, d\mu)$ converges to $\int_E x^*f\, d\mu$. Condition (a) and Vitali's theorem also insure that if $\lim_{n\to\infty} x_n^*f = x_0^*f$ a.e., then $\lim_{n\to\infty} \int_E x_n^*f\, d\mu = \int_E x_0^*f\, d\mu$. These conditions may be replaced by any others that similarly guarantee the convergence of the appropriate scalar integrals. In the light of the Phillips example, it is surprising indeed that this suffices to insure the integrability of f.

THEOREM 3. *Let (Ω, Σ, μ) be a finite perfect measure space and let $f \colon \Omega \to X$. If there is a sequence (f_n) of Pettis integrable functions from Ω into X such that*

(a) *The set $\{x^*f_n \colon x^* \in \mathrm{Ball}(X^*),\ n \in N\}$ is uniformly integrable, and*

(b) *for each x^* in X^* $\lim_{n\to\infty} x^*f_n = x^*f$ in measure,*

then f is Pettis integrable and $\lim_{n\to\infty}\int_E f_n\, d\mu = \int_E f\, d\mu$ weakly for each E in Σ.

PROOF. Fix E in Σ, and let C be the weak closure of $\{\int_E f_n\, d\mu \colon n \in N\}$. Since Vitali's convergence theorem guarantees that $\lim_{n\to\infty} \int_E x^*f_n\, d\mu = \int_E x^*f\, d\mu$ for each x^* in X^*, we observe that C is bounded and that $C \setminus \{\int_E f_n\, d\mu \colon n \in N\}$ contains at most one point. Suppose C is not weakly compact. An appeal to a theorem of James [4, Theorem 1] produces a bounded sequence (x_k^*) in X^*, a sequence (x_n) in C, and an $\varepsilon > 0$ such that $x_k^*(x_n) = 0$ for $k > n$ and $x_k^*(x_n) > \varepsilon$ for $n > k$. By passing to subsequences and relabeling, we can find a subsequence (g_n) of (f_m) and a subsequence (y_k^*) of (x_m^*) such that

$$\int_E y_k^* g_n\, d\mu = 0 \quad \text{for } k > n,$$

$$\int_E y_k^* g_n\, d\mu > \varepsilon \quad \text{for } n > k, \quad \text{and}$$

$$\lim_{n\to\infty} \int_E x^* g_n\, d\mu = \int_E x^*f\, d\mu \quad \text{for all } x^* \text{ in } X^*.$$

Apply Lemma 1 to the sequence $(y_k^* f)$ to find a subsequence $(y_{k_j}^* f)$ that converges almost everywhere. If y_0^* is any weak*-cluster point of (y_k^*) then $\lim_{j\to\infty} y_{k_j}^* f = y_0^* f$ a.e. Vitali's theorem now shows that $\lim_{j\to\infty} \int_E y_{k_j}^* f \, d\mu = \int_E y_0^* f \, d\mu$. To force a contradiction, note that for each k $\lim_{n\to\infty} \int_E y_k^* g_n \, d\mu = \int_E y_k^* f \, d\mu$. Hence $\int_E y_k^* f \, d\mu > \varepsilon$ for each k, and $\int_E y_0^* f \, d\mu \geqslant \varepsilon$. On the other hand, notice that since each g_n is Pettis integrable, the function $x^* \to \int_E x^* g_n \, d\mu$ is weak* continuous. Hence, if (y_α^*) is a subnet of (y_k^*) that converges weak* to y_0^*, then

$$\lim_\alpha \int_E y_\alpha^* g_n \, d\mu = \lim_\alpha y_\alpha^* \int_E g_n \, d\mu = y_0^* \int_E g_n \, d\mu = \int_E y_0^* g_n \, d\mu.$$

Since this holds for each n, and since $\lim_{n\to\infty} \int_E y_0^* g_n \, d\mu = \int_E y_0^* f \, d\mu$, we see that $\int_E y_0^* f \, d\mu = 0$. This contradicts the inequality $\int_E y_0^* f \, d\mu > \Sigma$, and proves that C is weakly compact. Since $\lim_{n\to\infty} \int_E x^* f_n \, d\mu = \int_E x^* f \, d\mu$, the sequence $(\int_E f_n \, d\mu)$ of Pettis integrals is weakly Cauchy. It follows from the weak compactness of C that $\lim_{n\to\infty} \int_E f_n \, d\mu$ exists weakly in X. This limit can only be $\int_E f \, d\mu$. As this holds for each E in Σ, the function f is Pettis integrable.

Theorem 3 can also be stated in the form of the traditional Dominated Convergence Theorem.

COROLLARY 4. *Let (Ω, Σ, μ) be a finite perfect measure space and let $f: \Omega \to X$. Suppose there is a sequence (f_n) of Pettis integrable functions from Ω into X such that $\lim_{n\to\infty} x^* f_n = x^* f$ a.e. for each x^* in X^* (the null set on which convergence fails may vary with x^*). If there is a scalar function ψ with $\|f_n(\cdot)\| < \psi(\cdot)$ a.e. for all n and if $\int \psi \, d\mu < \infty$, then f is Pettis integrable and $\lim_{n\to\infty} \int_E f_n \, d\mu = \int_E f \, d\mu$ weakly for each E in Σ.*

III. Simple functions and Pettis integrability. The type of convergence of Theorem 3 and Corollary 4, $\lim_{n\to\infty} x^* f_n = x^* f$ for each x^*, is the natural notion of convergence for weakly measurable functions. We will show that this convergence lies at the very heart of the Pettis integral. Consider the following example.

EXAMPLE 5. Let μ denote Lebesgue measure on $[0, 1]$ and define $f: [0, 1] \to L_\infty(\mu)$ by

$$f(t) = \chi_{[0, t)}.$$

This function is weakly measurable (each $x^* f$ is a function of bounded variation) and Pettis integrable, but not strongly measurable. We will show that f is the limit in the sense above of a sequence of simple functions.

Let π_n be the nth dyadic partition of $[0, 1]$. For any point t belonging to a dyadic interval $[(i - 1)/2^n, i/2^n)$ in π_n, let

$$f_n(t) = \chi_{[0, i/2^n)} \in L_\infty(\mu).$$

This defines a sequence (f_n) of simple functions from $[0, 1]$ into $L_\infty(\mu)$. Any element x^* of $L_\infty^*(\mu)$ may be identified with a bounded additive measure β that vanishes on sets of μ-measure 0. It follows from this identification that for t in $[(i - 1)/2^n, i/2^n)$

$$x^* f_n(t) = \beta([0, i/2^n))$$

and
$$x^*f(t) = \beta([0, t)).$$
Now, for fixed t, let $E_{t,n}$ be the element of π_n that contains t. We have
$$|x^*f(t) - x^*f_n(t)| \leqslant |\beta|(E_{t,n}).$$
It follows from the boundedness of β that $\lim_{n\to\infty} |\beta|(E_{t,n}) = 0$ for all but countably many t. We thus see that
$$\lim_{n\to\infty} x^*f_n = x^*f \quad \text{a.e.}$$
for each x^* in $L_\infty^*(\mu)$.

Example 5 is not an anomaly. We will now show that all Pettis integrable functions are limits in our sense of simple functions, just as all Bochner integrable functions are norm limits of simple functions. Our theorem completely characterizes Pettis integrability in terms of this convergence of simple functions. As Phillips's example shows, this result involves much more than weak measurability. The resulting juxtaposition of the conditions for Pettis integrability and Bochner integrability reveals the origin of the inherent complexity of the Pettis integral.

To prepare for the proof of Theorem 6, we establish the following notation. Let (Ω, Σ, μ) be a finite measure space. For any partition π of Ω into measurable sets we define the operator $E_\pi: L_1(\mu) \to L_1(\mu)$ by
$$E_\pi(\psi) = \sum_{E \in \pi} \chi_E(\cdot) \frac{\int_E \psi \, d\mu}{\mu(E)} .$$
This well-known operator maps each element of $L_1(\mu)$ into its conditional expectation relative to the σ-field generated by π. It is easily seen that $\|E_\pi\| \leqslant 1$. If we order the partitions by refinement, then for any ψ in $L_1(\mu)$ the net $(E_\pi \psi)$ converges to ψ in the $L_1(\mu)$ norm.

THEOREM 6. *Let (Ω, Σ, μ) be a finite perfect measure space, and let $f: \Omega \to X$. Then f is Pettis integrable if and only if there is a sequence (f_n) of simple functions from Ω into X such that*
   (a) *The set $\{x^*f_n: x^* \in \text{Ball}(X^*), n \in N\}$ is uniformly integrable, and*
   (b) *for each x^* in X^* $\lim_{n\to\infty} x^*f_n = x^*f$ a.e.*

PROOF. Since simple functions are Pettis integrable, one direction of this is immediate from Theorem 3. For the converse, suppose f is Pettis integrable and define $T: X^* \to L_1(\mu)$ by $T(x^*) = [x^*f]$. A brief computation shows that the adjoint of T acts on $L_\infty(\mu)$ by means of Pettis integration: $T^*(\varphi) = \int \varphi f \, d\mu$ for all φ in $L_\infty(\mu)$. According to Stegall's theorem the range of an indefinite Pettis integral is relatively norm compact [3, Proposition 3J]. It follows that T^* is a compact operator [1, Theorem 18, p. 161]; hence T is also compact. For each x^* in X^* the net $(E_\pi Tx^*)$ converges in the $L_1(\mu)$ norm to Tx^*, and since T is compact this convergence is uniform on $\text{Ball}(X^*)$. Thus E_π converges to T in the operator norm. Extract an increasing sequence (π_n) of partitions such that $\lim_{n\to\infty} \|E_{\pi_n} T - T\| = 0$ and define a sequence (f_n) of simple functions from Ω into X by
$$f_n(\cdot) = \sum_{E \in \pi_n} \chi_E(\cdot) \frac{\int_E f \, d\mu}{\mu(E)} .$$

Since $x^*f_n = E_{\pi_n} Tx^*$, we see that for each x^* in X^* $\lim_{n\to\infty} x^*f_n = x^*f$ in the $L_1(\mu)$ norm, and this convergence is uniform for x^* in Ball(X^*). Thus $\{x^*f_n \colon x^* \in$ Ball(X^*), $n \in N\}$ is uniformly integrable. For fixed x^* in X^* the sequence (x^*f_n) is an L_1-convergent martingale. Doob's Martingale Convergence Theorem now implies that $\lim_{n\to\infty} x^*f_n = x^*f$ a.e. for each x^* in X^*, and this completes the proof.

The nature of the convergence of (x^*f_n) to x^*f forms a large part of the distinction between the Bochner and Pettis integrals. It follows easily from Theorem 6 that a bounded function $f\colon \Omega \to X$ is Pettis integrable if and only if there is a bounded sequence (f_n) of simple functions such that $\lim_{n\to\infty} x^*f_n = x^*f$ a.e. for all x^* in X^*. The exceptional null set on which this convergence fails may vary with x^*. If there exists a fixed null set A such that $\lim_{n\to\infty} x^*f_n(\omega) = x^*f(\omega)$ for all x^* in X^* and for all ω not in A, then it follows from the Hahn-Banach theorem that f has separable range off A; hence f is strongly measurable and Bochner integrable. The distinction between the Bochner integral and the Pettis integral is thus the distinction between stationary exceptional sets and mobile exceptional sets.

Our final theorem summarizes this discussion.

THEOREM 7. *Let (Ω, Σ, μ) be a finite perfect measure space and let f be a bounded function from Ω into X. Then*

(a) *The function f is Pettis integrable if and only if there is a bounded sequence (f_n) of simple functions from Ω into X with $\lim_{n\to\infty} x^*f_n = x^*f$ a.e. for all x^* in X^*.*

(b) *The function f is Bochner integrable if and only if there is a null set A and a sequence (f_n) of simple functions from Ω into X with $\lim_{n\to\infty} x^*f_n(\omega) = x^*f(\omega)$ for all x^* in X^* and for all ω not in A.*

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