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MINIMAL POSITIVE 2-SPANNING SETS OF VECTORS[1]

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ABSTRACT. Let $(v_1, \ldots, v_n)$ be a sequence in an m-dimensional vector space V over an ordered field such that, for each i, $\{v_j : j \neq i\}$ positively spans V. It is shown that if $(v_1, \ldots, v_n)$ is minimal with this property, then

$$n < \begin{cases} 4m & \text{if } m < 5 \\ m(m+1)/2 + 5 & \text{if } m > 5 \end{cases}$$

and all cases are determined in which $n = 4m$, $m < 4$. An application to convex polytopes is given.

1. Introduction. Let V be a vector space over an ordered field K. A *positive spanning set* (henceforth abbreviated as PSS) for V is a set $\{v_1, \ldots, v_n\}$ of vectors in V such that each vector in V is a linear combination of the v_i with nonnegative coefficients. Equivalently, every open half-space in V (one side of a hyperplane) contains some v_i. This equivalence can be seen by considering the positive span of the v_i, which is a convex cone with apex at the origin. Either this cone is the entire space or else it lies on one side of a hyperplane.

A positive k-spanning set (PkSS) is a sequence $(v_1, \ldots, v_n)$ of vectors in V, not necessarily distinct, such that, for each set I of $k-1$ indices $i_1, \ldots, i_{k-1}$, $\{v_j : j \notin I\}$ is a PSS. Equivalently, every open half-space in V contains v_i for at least k values of i. We are concerned with minimal PkSS's, by which we mean a PkSS such that if any v_i is removed, the remaining $v_j, j \neq i$, do not form a PkSS.

It is well known that every minimal PSS for K^m contains at most $2m$ vectors; moreover such a set with exactly $2m$ vectors necessarily consists of a basis for K^m and negative multiples of that basis [2, Theorem 6.7]. When $k > 1$, it is tempting to conjecture that the "worst case" (the most vectors) occurs when the PkSS consists of $2km$ vectors, k of them in each direction along m linearly independent lines through the origin. Equivalently, the PkSS consists of scalar multiples of some basis.

CONJECTURE 1. Every minimal PkSS for K^m contains at most $2km$ vectors, and every minimal PkSS having exactly $2km$ vectors consists of scalar multiples of some basis for K^m.

The case $m = 1$ is trivial. We will prove

THEOREM 1. *Conjecture* 1 *holds when* $m = 2$, *for all* $k > 1$.

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This will be established by a straightforward argument. When $m \geq 3$, however, that argument breaks down at a crucial point. By a more subtle method we will prove the following two theorems, which are the main results of this paper:

THEOREM 2. *Conjecture* 1 *holds when* $k = 2$ *and* $m \leq 4$.

THEOREM 3. *If* $m \geq 5$, *every* $P2SS$ *for* K^m *contains at most* $m(m + 1)/2 + 5$ *vectors*.

Theorems 2 and 3 imply

COROLLARY 1. *When* $m \leq 5$, *every minimal* $P2SS$ *for* K^m *contains at most* $4m$ *vectors*.

Considering the case of two vectors in each direction along m independent lines through the origin, we see that the bound in Corollary 1 is best possible.

2. Positive k-spanning sets in combinatorics. There are at least two ways in which positive k-spanning sets arise in connection with combinatorial structures. The first (and simpler) of these occurs in the incidence matrix of a digraph (a graph with directed edges), where the element in row i and column j is

$$a_{ij} = \begin{cases} 1 & \text{if edge } j \text{ terminates at vertex } i, \\ -1 & \text{if edge } j \text{ originates at vertex } i, \\ 0 & \text{if otherwise.} \end{cases}$$

Clearly each column is in the hyperplane H defined by the equation $\Sigma x_i = 0$. It can be shown [5] that the columns form a PkSS for H if and only if the digraph is *k-edge-connected*. This means that it is connected (there is a directed path from any vertex to any other vertex) and remains so after removal of any $k - 1$ edges. Such a digraph is *minimally k-edge-connected* if its k-edge connectivity is destroyed by removal of any edge. In this context Conjecture 1 reduces to the statement that every minimally k-edge-connected digraph with $m + 1$ vertices has at most $2km$ edges, and that if it has exactly $2km$ edges then it can be constructed by replacing each edge of a tree with $2k$ edges, k in each direction. This special case of Conjecture 1 has been proven by Dalmazzo in [1].

Positive k-spanning sets, for $k \geq 2$, also occur in connection with convex polytopes. To each d-dimensional convex polytope P with vertices $P_1, \ldots, P_n$, there is associated a sequence of vectors $(v_1, \ldots, v_n)$, called the *Gale diagram* of P, in a real vector space $\mathbf{R}^m$ of dimension $m = n - d - 1$. The fundamental property of Gale diagrams is this:

THEOREM. *For each index set I, a set $\{P_i: i \in I\}$ is the vertex set of some face F of P if and only if $\{v_i: i \notin I\}$ is a PSS for its linear span, and F is a simplex if and only if $\{v_i: i \notin I\}$ is a PSS for $\mathbf{R}^m$.* (*See* [3, §5.4], *or* [4] *or* [6].)

It follows from this (see [3] or [4]) that the Gale diagram of P is always a P2SS for $\mathbf{R}^m$, and it is a PkSS, for $k \geq 3$, if and only if P is $(k - 1)$-*neighborly*. This means that every set of $k - 1$ vertices is the vertex set of a face of P. Necessarily such a face is a simplex. In view of this, Conjecture 1 implies

CONJECTURE 2. Let P be an r-neighborly convex polytope of dimension d, and suppose that P has fewer than $(1 + 1/(2r + 1))(d + 1)$ vertices. Then P contains a vertex v_0 such that, for each set of r vertices $v_1, \ldots, v_r$ different from v_0, the set $\{v_0, \ldots, v_r\}$ is the vertex set of a face (necessarily a simplex) of P.

The validity of Conjecture 1 for $m \leq 2$ implies that of Conjecture 2 whenever P has at most $d + 3$ vertices. When $r = 1$, Conjecture 2 reduces to

CONJECTURE 3. If P is a convex polytope of dimension d, and if P has fewer than $4(d + 1)/3$ vertices, then P contains a vertex which is connected to all other vertices of P by edges.

Corollary 1 to Theorems 2 and 3 implies that Conjecture 3 is true for all $d \leq 17$ and also for $d = 20$. Whenever $n \leq d + 6$, the number of such vertices is at least $4d - 3n + 4$. Moreover if $n \leq d + 5$ and if $4d - 3n + 4 = t$ is the exact number of such vertices, then the Gale diagram of P consists of a minimal P2SS of $4m$ points in $\mathbf{R}^m$, $m \leq 4$, along with t additional points at the origin. Then P is a t-fold pyramid on the free join of $n - d - 1$ convex quadrilaterals.

3. Proofs of theorems. Before proceeding with the proofs of Theorems 1, 2 and 3, we consider the analog of positive k-spanning sets in standard linear algebra. Dropping the assumption that K is an ordered field and the requirement that coefficients be nonnegative, we define a *k-spanning set* of vectors in a K-vector space to be a sequence of n vectors, any $n - k + 1$ of which form a spanning set for the space. The analog of Conjecture 1 states that every minimal k-spanning set for K^m contains at most km vectors, and that a minimal k-spanning set with exactly km vectors consists of scalar multiples of a basis. This can be established by an induction argument as follows:

Obviously the result is true for $k = 1$. Proceeding by induction on k, assume it holds for minimal $(k - 1)$-spanning sets and consider a minimal k-spanning set $(v_1, \ldots, v_n)$. Assume without loss of generality that $(v_1, \ldots, v_{n_0})$ is a minimal $(k - 1)$-spanning set for K^m. Then $n_0 \leq (k - 1)m$. We claim that the vectors $v_{n_0+1}, \ldots, v_n$ are linearly independent. Suppose one of these (say v_n) is a linear combination of the others. Then we will show that $(v_1, \ldots, v_{n-1})$ is a k-spanning set, contrary to assumption. To establish this, consider any set I of $k - 1$ indices less than n. If at least one of these indices is greater than n_0, then the v_i, $i \notin I$, $i \leq n_0$, form a spanning set for K^m. Consequently so do all v_i, $i \notin I$, $i < n$. In the other case, if all members of I are less than or equal to n_0, consider the v_i, $i \notin I$, $i \leq n$. These form a spanning set for K^m since all v_i form a k-spanning set, and their span is the same as that of the v_i, $i \notin I$, $i < n$.

Thus the v_i, $i > n_0$, are linearly independent. It follows that $n \leq km$. Finally, if $n = km$, then $n_0 = (k - 1)m$ so by inductive hypothesis the v_i, $i \leq n_0$, are all scalar multiples of some basis. Also, the m vectors v_i, $i > n_0$, form a basis. Finally, it is easy to see that these bases must be scalar multiples of each other.

We now consider the extent to which this proof can be adapted to positive k-spanning sets over an ordered field. We know Conjecture 1 is true for $k = 1$. In the inductive step, we assume that $(v_1, \ldots, v_n)$ is a minimal PkSS for K^m and that

$(v_1, \ldots, v_{n_0})$ is a minimal P$(k-1)$SS. By the same argument as above, we see that the vectors $v_{n_0+1}, \ldots, v_n$ are *positively independent*, which means that none of them is a positive combination of the others. The proof breaks down, however, when we attempt to conclude that $n - n_0 < 2m$. The problem is that when $m > 3$, positively independent sets can be arbitrarily large: Consider, for example, the vertices of a convex n-gon in $\mathbf{R}^3$, lying in a plane which does not pass through the origin.

When $m = 2$, however, a positively independent set can contain no more than four vectors, and if it contains exactly four then they are scalar multiples of a basis. To see this, let S be a positively independent set of vectors in K^2 and let u and v be two linearly independent vectors in S. If all members of S are scalar multiples of u and v, then it is easy to see that the assertion is true. If, on the other hand, S contains some vector w which is not a scalar mutiple of u or v, then there is a linear dependence $au + bv + cw = 0$, in which a, b and c are all nonzero. If a, b and c are all positive or all negative, then every vector in K^2 is a positive combination of u, v and w, so S contains only three vectors. On the other hand, if a, b and c are not all of the same sign, then one of the vectors u, v, w is a positive combination of the other two, contrary to the assumption that S is positively independent.

We have proved Theorem 1.

PROOF OF THEOREM 2. Theorem 2 holds for $m = 1$ trivially. Proceeding by induction on m, fix $m = 2, 3$ or 4 and assume the result for all minimal positive 2-spanning sets in dimension $m - 1$. Let $(v_1, \ldots, v_n)$ be a minimal P2SS for K^m. Note that for each i, there exists some $j \neq i$ such that v_i and v_j are alone in some open half-space (i.e., no v_k, $k \neq i, j$, is in this open half-space). If this were not true, then deletion of v_i would yield a smaller P2SS.

At this point we split the proof into two cases.

Easy case. Suppose two of the v_i (say v_1 and v_2) are on the same ray from the origin. Equivalently, they are positive multiples of each other. Then there exists a hyperplane H, one side of which contains no v_i, $i > 3$. Factoring out the subspace spanned by v_1, we project all v_i, $i > 3$, on H. Let v_i go to $h_i \in H$. $(h_3, \ldots, h_n)$ is a P2SS for H, so at most $4(m-1)$ of the h_i form a minimal P2SS for H. Consider the corresponding v_i: If at least two of these are not in H, then these v_i (at most $4(m-1)$ vectors) along with v_1 and v_2 form a P2SS. In any case, at most two other v_i need be added to this set so that at least two v_i are on the opposite side of H from v_1, and this results in a P2SS containing at most $4m$ vectors. Moreover the characterization of a minimal P2SS with $4(m-1)$ vectors in H leads to the corresponding result in $\mathbf{R}^m$: Assuming that $n = 4m$, it must be the case that exactly $4(m-1)$ of the h_i (say all h_i, $i > 4$) form a minimal P2SS for H, in which case they must all be scalar multiples of a basis for H. Moreover none of the v_i, $i > 4$, can be on the opposite side of H from v_1, since v_3 and v_4 are both necessary for a P2SS for $\mathbf{R}^m$. It follows that $v_i = h_i$ for all $i > 4$, and it remains only to show that v_3 and v_4 are both scalar multiples of v_1. Equivalently, h_3 and h_4 are both 0. But this is clear, since for example if $h_3 \neq 0$, then (in view of the special nature of the h_i, $i > 4$) h_3 could have been chosen as part of a minimal P2SS for H in the argument above, leading to a P2SS for $\mathbf{R}^m$ with fewer than $4m$ vectors.

That completes the induction step in the easy case.

Hard case. Suppose no two of the v_i are on the same ray from the origin. We introduce a graph G having vertices $1, \ldots, n$, in which $\{i, j\}$ is an edge if v_i and v_j appear alone in some open half-space. Define, for each i, $A_i = \{j: \{i, j\}$ is an edge$\}$. For any i, we change the original P2SS as follows:

remove v_i;

"double" all vectors $v_j, j \in A_i$; that is, introduce another copy of v_j.

Clearly, we still have a P2SS; reduce it to a minimal P2SS. It is easy to see that if a vector v_k is removed, and if $j \in A_k$, then v_j must have been one of the vectors that was doubled. Consequently $j \in A_i$. In other words, k is in the set $B_i = \{k: A_k \subseteq A_i\}$. Note also that $i \in B_i$ trivially. Thus a new minimal P2SS has been formed by doubling all v_i, $i \in A_i$, and removing certain of the v_k, $k \in B_i$ (although not necessarily all such v_k).

Suppose $|B_i| < |A_i|$. Then the new minimal P2SS contains at least n vectors, which are not all distinct. By the easy case, we conclude that $n \leqslant 4m$.

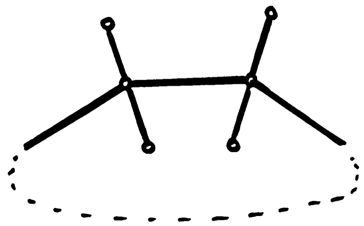
Thus if G contains a vertex i such that $|B_i| < |A_i|$, we are finished. We will show that this is the case if $m \leqslant 3$, and that if $m = 4$ all "bad" graphs can be determined and dealt with by another argument.

LEMMA 1. *For each i, the $v_j, j \in A_i$, are linearly independent.*

PROOF. v_i is a positive combination of the v_k, $k \neq i$; hence by a well-known argument of Carathéodory (see [3, Theorem 2.3.5]) it follows that v_i is a positive combination of a linearly independent subset of $\{v_k: k \neq i\}$. Clearly all $v_j, j \in A_i$, must be in this subset.

Thus each vertex of G has degree at most m. Moreover from the definition of G it is clear that G contains no isolated vertices (vertices of degree 0). Using only these properties, we can consolidate most of the remainder of the proof into a strictly graph-theoretic lemma:

LEMMA 2. *Let G be a graph with no isolated vertices and let d be the maximum degree of all vertices in G. If $d \leqslant 3$, then G contains a vertex i for which $|B_i| < |A_i|$. If $d = 4$ and $|B_i| > |A_i|$ for each vertex i, then each connected component of G consists of a circuit with two extra edges at each vertex*:



Before proceeding with the proof of Lemma 2, we establish another lemma:

LEMMA 3. *If $A_j \subseteq B_i$, then $B_j \subseteq A_i$.*

Proof. If $b \in B_j$, then $A_b \subseteq A_j \subseteq B_i$. Fix any $a \in A_b$ (G contains no isolated vertices); then $b \in A_a \subseteq A_i$.

Proof of Lemma 2. Assume that $|B_i| > |A_i|$ for each vertex i. First we show that $d \geq 4$. Fix a vertex i of degree d. Let $A_i = \{a_1, \ldots, a_d\}$, $B_i = \{b_1, \ldots, b_e = i\}$, and let $d_1, \ldots, d_e$ denote the degrees of $b_1, \ldots, b_e$. We can assume $d_1 \leq d_2 \leq \cdots \leq d_e = d$. For each $b \in B_{b_1}$, we have $A_b \subseteq A_{b_1} \subseteq A_i$; hence $b \in B_i$. Then $b = b_k$ for some k. Moreover $|A_{b_k}| = d_k \geq d_1 = |A_{b_1}|$, implying that $A_{b_k} = A_{b_1}$. Thus without loss of generality we can assume that $A_{b_1} = A_{b_2} = \cdots = A_{b_r}$, where $r = |B_{b_1}| > d_1$. Then any $j \in A_{b_1}$ is adjacent to $b_1, \ldots, b_r$, implying $r \leq d$. It follows that $r < e$; hence $i \neq b_1, \ldots, b_r$. But every $j \in A_{b_1}$ is adjacent to i, so in fact $r < d$. Moreover, if $r = d - 1$, then $A_j = \{b_1, \ldots, b_r, i\} \subseteq B_i$; hence by Lemma 3, $B_j \subseteq A_i$. But then $|B_j| \leq |A_i| = d = r + 1 = |A_j|$, contrary to assumption. Thus we have $1 \leq d_1 < r \leq d - 2$, implying that $d \geq 4$.

Now assume that $d = 4$. Continuing the argument above, we must have $d_1 = 1$ and $r = 2$. We claim that of the four vertices adjacent to i, two have degree 1 and two have degree 4.

Let $A_{b_1} = A_{b_2} = \{a\}$. We can assume $a = a_1$.

$$\begin{array}{ccccc} b_1 & & a_2 & & \\ | & & | & & \\ a & — & i & — & a_4 \\ | & & | & & \\ b_2 & & a_3 & & \end{array}$$

If a has degree 3, then $A_a \subseteq B_i$; hence by Lemma 3, $B_a \subseteq A_i$. Since $|B_a| > |A_a| = 3$, we have $B_a = A_i$. Again by Lemma 3, $B_i \subseteq A_a$. But then $|B_i| < |A_i|$. We conclude that a has degree 4.

Let $A_a = \{b_1, b_2, i, j\}$. $|A_j| \leq 4$ and $|B_a| > |A_a| = 4$, so $B_a - A_j$ contains some k. $A_k \subseteq A_a$, but $j \notin A_k$. In particular, $k \neq a$, so $b_1, b_2 \notin A_k$. Thus $A_k = \{i\}$ and we can assume that $k = a_2$. k has degree 1, so $|B_k| \geq 2$. Each member of B_k is adjacent only to i, so we can assume that a_3 has degree 1.

It remains to show that a_4 has degree 4. $|B_i| > |A_i| = 4 = |A_a|$, so $B_i - A_a$ contains some b. Necessarily $A_b = \{a_4\}$. The argument above can then be repeated with b_1 replaced by b, yielding the fact that a_4 has degree 4.

Thus, as claimed, i is adjacent to two vertices of degree 1 and two vertices of degree 4. Since i was an arbitrary vertex of degree 4, we conclude that every vertex of degree 4 is adjacent to two vertices of degree 1 and two vertices of degree 4. Moreover the argument shows that each connected component of G contains a vertex of degree 4, which is the maximum degree in G. (Otherwise, by applying the argument to a single component having no vertex of degree 4, we would obtain a contradiction.) Finally, then, it follows that each component consists of a circuit with two extra edges at each vertex, as claimed. Lemma 2 is now proved.

We now resume the proof of Theorem 2. Lemma 1 shows that the maximum degree d of vertices in G is at most m, which is at most 4. Thus, by Lemma 2, G contains a vertex i for which $|B_i| \leq |A_i|$ except in the case in which each component of G consists of a circuit with two extra edges at each vertex. In that case, either G contains at most 15 vertices, or else G contains two nonadjacent vertices i, j of degree 4 such that A_i and A_j are disjoint. In the latter case, consider the set $\{v_k : k \neq i, j\}$. This is a PSS for K^4; hence it contains a minimal PSS. The latter necessarily contains $A_i \cup A_j$. A minimal PSS with at least eight vectors consists of a basis for K^4 and negative multiples of that basis; hence $\{v_1, \ldots, v_n\}$ contains eight such vectors. It then follows (see Lemma 4) that $n \leq 16$, and that $n = 16$ if and only if all v_i are scalar multiples of the basis.

LEMMA 4. *Let $(v_1, \ldots, v_n)$ be a minimal P2SS for K^m containing a basis for K^m and negative multiples of that basis. Then $n \leq 4m$, and if $n = 4m$ then all v_i are scalar multiples of that basis.*

PROOF. Without loss of generality we can assume that the basis is $(v_1, \ldots, v_m)$ and that $v_{m+1}, \ldots, v_{2m}$ are negative multiples of this basis. For each $i > 2m$ let $v_i = a_{i1}v_1 + \cdots + a_{im}v_m$. Since $(v_1, \ldots, v_n)$ is a P2SS, for each $j = 1, \ldots, m$ there must exist $i > 2m$ such that $a_{ij} > 0$ and $i > 2m$ such that $a_{ij} < 0$. Selecting two such values of i for each j, and including $v_1, \ldots, v_{2m}$, we obtain a P2SS consisting of at most $4m$ vectors. Thus $n \leq 4m$. If any v_i is not a scalar multiple of one of the basis vectors, then it can be chosen for more than one value of j, and consequently the P2SS contains fewer than $4m$ vectors.

PROOF OF THEOREM 3. The proof of Theorem 3 is accomplished by induction on m, using an adaptation of the argument in the proof of Theorem 2. Starting with $m = 5$ and assuming that $(v_1, \ldots, v_n)$ is a minimal P2SS for K^5, we consider the same two cases as in Theorem 2:

Easy case. If two v_i are on the same ray from the origin, then the argument employed in the easy case of Theorem 2, along with the result of Theorem 2 for $m = 4$, implies that $n \leq 20$. Moreover it shows that if $n = 20$ and two v_i are on the same ray from the origin, then all v_i are scalar multiples of some basis.

Hard case. If no two v_i are on the same ray from the origin, form the graph G as in the proof of Theorem 2. Define A_i and B_i for each i as before. For any i, change the P2SS as before. Suppose, for some i, we have $|B_i| \leq |A_i| + 1$. Then the new minimal P2SS contains at least $n - 1$ vectors. But the vectors of this set are not all distinct, so by the easy case we have $n - 1 \leq 20$; moreover if $n - 1 = 20$, then the new set consists of scalar multiples of some basis. In this latter case, we conclude that the original P2SS $(v_1, \ldots, v_n)$ contains a basis and negative multiples of that basis. But then, by Lemma 4, $n \leq 20$.

Thus in all cases, $n \leq 20$ when $m = 5$, provided that G contains a vertex i for which $|B_i| \leq |A_i| + 1$. We will show that such a vertex exists, except in one special case which will be handled separately. More generally, we will establish

LEMMA 5. *Let G be a graph with no isolated vertices and let d be the maximum degree of all vertices in G. If $d \geq 4$, then either G contains a vertex i for which $|B_i| \leq |A_i| + d - 4$, or else each connected component of G consists of a circuit with $d - 2$ extra edges at each vertex.*

PROOF. Assume that $|B_i| > |A_i| + d - 4$ for each i. As in the proof of Lemma 2, fix a vertex i of degree d and set $A_i = \{a_1, \ldots, a_d\}$, $B_i = \{b_1, \ldots, b_e = i\}$. As in the proof of Lemma 2, we can assume that $A_{b_1} = A_{b_2} = \cdots = A_{b_r}$ where $r = |B_{b_1}|$. By the same argument as in Lemma 2, we obtain $r \leqslant d - 2$. Moreover we have $r = |B_{b_1}| > |A_{b_1}| + d - 4 = d_1 + d - 4$ where d_1 is the degree of b_1. Since $d_1 \geqslant 1$, we obtain $r \geqslant d - 2$. Thus $r = d - 2$, and it follows that $d_1 = 1$.

We claim that i is adjacent to two vertices of degree d, and to $d - 2$ vertices of degree 1.

Let $A_{b_1} = \cdots = A_{b_r} = \{a\}$ and without loss of generality assume $a = a_1$. The set A_a includes $i, b_1, \ldots, b_r$ and at most one other vertex, since $r = d - 2$. If no other vertex is adjacent to a, then a contradiction results as in the proof of Theorem 2. (The only change here is that $|A_a| = d - 1$.) We conclude that $A_a = \{b_1, \ldots, b_r, i, j\}$ for some vertex j. Thus $a = a_1$ has degree d.

As in the proof of Theorem 2, $B_a - A_j$ contains some vertex k, which can be shown to be adjacent only to i. Moreover the same is true for all members of B_k, of which there must be at least $|A_k| + d - 3 = d - 2$. Thus, without loss of generality, we can assume that $a_2, a_3, \ldots, a_{d-1}$ all have degree 1.

It remains to show that a_d has degree d. We have $|B_i| > |A_i| + d - 4 = 2d - 4 \geqslant d \geqslant |A_a|$, so $B_i - A_a$ contains some b. Necessarily $A_b = \{a_d\}$. The argument above can then be repeated with b_1 replaced by b, yielding the fact that a_d has degree d.

Thus i, an arbitrary vertex of degree d, is adjacent to two vertices of degree d and to $d - 2$ vertices of degree 1. As in the proof of Lemma 2, the characterization of G then follows easily. That completes the proof of Lemma 5.

Returning to the proof of Theorem 3, we apply Lemma 5 to complete the case $m = 5$, except when each component of G consists of a circuit with $d - 2$ extra edges at each vertex. If $d < 5$, then $|B_i| \leqslant |A_i| + 1$ for each i, so the only remaining case that must be considered is when $d = 5$. In that case, either $n \leqslant 20$ or else G contains nonadjacent vertices i and j of degree 5 such that A_i and A_j are disjoint. In the latter case, the proof is completed as in Theorem 2, using Lemma 4.

Finally, we establish the induction step when $m > 5$. Lemma 5 implies that either G contains a vertex i for which $|B_i| \leqslant |A_i| + m - 4$, or else each component consists of a circuit with $m - 2$ extra edges at each vertex. In the former case, the vertex i can be used for the induction step. In the latter case, either $n \leqslant 5(m - 1) \leqslant m(m + 1)/2 + 5$, or else G contains nonadjacent vertices i and j of degree m with A_i and A_j disjoint, and the proof is completed as before.

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