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# THE $L^2$-NORM OF MAASS WAVE FUNCTIONS

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ABSTRACT. Let $D$ denote the fundamental domain for the full modular group. Suppose that $f \in L^2(D)$ satisfies the wave equation $\Delta f = \lambda f$, where $\Delta$ is the noneuclidean Laplacian, and further, assume that $f$ is a common eigenfunction for all the Hecke operators. Then upper and lower bounds for the $L^2$-norm of $f$ are determined which depend only on $\lambda$ and the first Fourier coefficient of $f$.

Let $D$ denote the standard fundamental domain for the action of the discrete group $\Gamma = SL_2(\mathbf{Z})$ on the upper half-plane $\mathbf{H}$, the action being given by $\sigma z = (az + b)/(cz + d)$ for all $\sigma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ and $z = x + iy \in \mathbf{H}$. The upper half-plane is equipped with a $\Gamma$-invariant noneuclidean metric $ds^2 = y^{-2}(dx^2 + dy^2)$ which induces the $\Gamma$-invariant measure $d\mu(z) = y^{-2}dxdy$ and the $\Gamma$-invariant Laplacian $\Delta = -y^2(\partial^2/\partial x^2 + \partial^2/\partial y^2)$. Let $L^2(D)$ denote the Hilbert space of square-integrable $\Gamma$-invariant functions on $D$ with respect to the measure $d\mu$, i.e., $f \in L^2(D)$ iff $f \circ \sigma = f$ for all $\sigma \in \Gamma$ and $\|f\| = \sqrt{(f, f)} < \infty$, where the inner product is defined by

$$(f, g) = \int_D f(z)\overline{g(z)}\, d\mu(z).$$

The operator $\Delta$ extends to a selfadjoint operator on $L^2(D)$ which we also denote by $\Delta$.

Following Maass [1], we consider functions $f \in L^2(D)$ which are solutions of the noneuclidean wave equation (which we may assume are real since $\Delta$ is selfadjoint) $\Delta f = \lambda f$, where $\lambda = \frac{1}{4} + r^2$ for some $r \in \mathbf{C}$; such an $f$ is called a Maass wave function for $\Gamma$. For the group $\Gamma = SL_2(\mathbf{Z})$, we know that $\lambda > 3\pi^2/2$ (cf. Roelcke [6, p. 208]) so that we may assume $r > 0$. Since $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma$, $f$ has a Fourier expansion of the form (cf. Maass [1, Theorem 5])

$$f(z) = \sum_{m \neq 0} c(m) y^{1/2} K_{ir}(2\pi |m| y) e^{2\pi i m x}$$

where $K_\nu$ is the modified Bessel function of the second kind; we shall call $c(m) = c_f(m)$ the $m$th Fourier coefficient of $f$.

For any function $f$ defined on $\mathbf{H}$ and for any integer $n \geq 1$, we define the Hecke operator $T(n)$ by (cf. Maass [1, p. 178])

$$(T(n)f)(z) = \frac{1}{\sqrt{n}} \sum_{\substack{ad = n \\ b \bmod d}} f\left(\frac{az + b}{d}\right) \qquad (d > 0).$$

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Then $T(n)$ acts on $L^2(D)$ as a selfadjoint operator for all $n \geqslant 1$. If $f$ is a Maass wave function which is a common eigenfunction for all the Hecke operators, i.e., if $T(n)f = \lambda(n)f$ for all $n \geqslant 1$, then Maass showed that (cf. [1, p. 180])

$$\lambda(n)c(m) = \sum_{d|(m,n)} c\left(\frac{mn}{d^2}\right) \tag{1}$$

for all $n \geqslant 1$ and $m \neq 0$, whence $c(n) = \lambda(n)c(1)$ for all $n \geqslant 1$. Since $f$ is real, then $c(-m) = \overline{c(m)}$ whence $c(1) \neq 0$ if $f \neq 0$. Consequently, if we normalize $f$ so that $c(1) = 1$, the Fourier coefficients of $f$ are just the eigenvalues of the Hecke operators, i.e., $c(m) = \lambda(|m|)$.

On the other hand, one frequently wishes to normalize $L^2$-functions to have norm 1. Clearly, these two normalizations cannot be achieved simultaneously in general for the functions we are considering. The purpose of this paper is to attempt to clarify this obscurity. Finally, I am grateful to Dr. S. J. Patterson for the suggestion that his paper [3] might be useful in studying the $L^2$-norm of these functions.

Our main result is the following

THEOREM. *Suppose $f \in L^2(D)$ is a Maass wave function satisfying $\Delta f = \lambda f$ with $\lambda = \frac{1}{4} + r^2$, $r > 0$. Furthermore, assume that $f$ is a common eigenfunction for all the Hecke operators. Then the $L^2$-norm of $f$ satisfies*

$$\frac{1}{6}|\Gamma(ir)|\,|c_f(1)| \leqslant \|f\| \leqslant \frac{1}{2}\left(\frac{5\alpha - 3}{5\alpha - 8}\right)^2 \left|\Gamma\left(\frac{\alpha}{2} + ir\right)\right|\,|c_f(1)|,$$

*the lower bound holding uniformly for $r \geqslant 200$ while the upper bound holds uniformly in $r > 0$ and $8/5 < \alpha \leqslant 13/5$.*

The main significance of this result is seen in the following (with $\alpha = 2$, say)

COROLLARY. *Suppose that $f$ satisfies the conditions of the above theorem with $\|f\| = 1$. Then the first Fourier coefficient of $f$ satisfies the inequality*

$$\frac{1}{7}|\Gamma(1 + ir)|^{-1} \leqslant |c_f(1)| \leqslant 6|\Gamma(ir)|^{-1}$$

*for all $r \geqslant 200$. In particular, Stirling's formula shows that $|c_f(1)|$ grows exponentially with $\sqrt{\lambda}$ as $\lambda \to \infty$.*

To prove the theorem, it suffices to assume that $|c_f(1)| = 1$. We begin by establishing the upper bound for $\|f\|$. Since $\operatorname{Im} z \geqslant y_0 = \sqrt{3}/2$ for all $z \in D$, it follows that for any $\alpha > 0$,

$$\|f\|^2 \leqslant y_0^{-\alpha} \int_S |f(z)|^2 y^\alpha \, d\mu(z) \tag{2}$$

where $S = \{z \in \mathbf{H}: |\operatorname{Re} z| \leqslant \frac{1}{2}\}$. For $\alpha$ sufficiently large, the integral in (2) can be evaluated and gives (cf. Moreno [2, p. 132])

$$\|f\|^2 \leqslant \frac{1}{4}(4\pi y_0)^{-\alpha} B\left(\frac{\alpha}{2}, \frac{\alpha}{2}\right) \left|\Gamma\left(\frac{\alpha}{2} + ir\right)\right|^2 \sum_{n \geqslant 1} |c(n)|^2 n^{-\alpha} \tag{3}$$

where $B(\cdot, \cdot)$ denotes the beta function. Rankin's convolution method implies the existence of a constant $A > 0$, possibly depending on $f$, such that $|c(n)| \leq An^{\beta}$ for all $n \geq 1$ for some $\beta$ satisfying $0 \leq \beta \leq 3/10$ (cf. Moreno [2, p. 143]). From this bound for $c(n)$, it follows that (cf. Rankin [5]; in fact, the argument is the same as Rankin's proof of Lemma 1 in view of (1))

$$|c(n)| \leq n^{\beta} d(n) \tag{4}$$

for all $n \geq 1$, where $d(n)$ denotes the number of positive divisors of $n$. Combining (3) and (4), and using the fact that $B(x, x) \leq \pi$ for $x \geq \frac{1}{2}$, it follows that

$$\|f\|^2 \leq \frac{1}{16y_0}\left|\Gamma\left(\frac{\alpha}{2} + ir\right)\right|^2 \sum_{n \geq 1} d^2(n) n^{-(\alpha - 2\beta)}$$

provided $\alpha > 2\beta + 1$, which may be rewritten as (cf. Ramanujan [4, p. 133])

$$\|f\| \leq \frac{1}{4} y_0^{-1/2}\left|\Gamma\left(\frac{\alpha}{2} + ir\right)\right| \zeta^2(\alpha - 2\beta)/\sqrt{\zeta(2\alpha - 4\beta)} \tag{5}$$

where $\zeta(s)$ denotes the Riemann $\zeta$-function. For any $a > 1$, we know that

$$1/(a-1) < \zeta(a) < a/(a-1)$$

from which the upper bound immediately follows by (5) if we require $\alpha$ to satisfy $1 < \alpha - 2\beta \leq 2$ with $\beta = 3/10$.

Next we must establish the lower bound for $\|f\|$. If $S = \{z \in \mathbf{H}: |\mathrm{Re}\, z| \leq \frac{1}{2}$ and $\mathrm{Im}\, z \geq 1\}$, then clearly $S \subseteq D$ so that $\|f\|^2 \geq \int_S |f(z)|^2\, d\mu(z)$ which by Parseval's theorem implies

$$\|f\|^2 \geq \int_1^{\infty} \sum_{m \neq 0} c(m)^2 K_{ir}^2(2\pi|m|y)\, d^{\times}y$$

where $d^{\times}y = y^{-1}dy$. Since $c(1) = c(-1) = 1$, then $\|f\|^2 \geq 2\int_1^2 K_{ir}^2(2\pi y)\, d^{\times}y$. By Lemma 4 of Patterson [3], we know that for any $0 < a < b$, then $(r > 0)$

$$\int_a^b K_{ir}^2(y)\, d^{\times}y \geq \left\{\left(\frac{3}{2} - \exp^{(b^2/2r)}\right)\log(b/a) - \frac{1}{2r}\right\}|\Gamma(ir)|^2,$$

whence

$$\|f\|^2 \geq \left\{2\left(\frac{3}{2} - \exp^{(8\pi^2/r)}\right)\log 2 - \frac{1}{r}\right\}|\Gamma(ir)|^2. \tag{6}$$

For $r \geq 200$, the expression in the bracket on the right-hand side of (6) is $\geq 0.028138\ldots > 1/36$, which completes the proof of the theorem.

REMARK. If the Ramanujan-Petersson conjecture for Maass wave functions is true for the group $\Gamma = SL_2(\mathbf{Z})$ (it is certainly true on the average by Moreno [2]), then we can take $\beta = 0$ in (5) so that the upper bound given above could be replaced by

$$\|f\| \leq \frac{1}{2}\left(\frac{\alpha}{\alpha - 1}\right)^2\left|\Gamma\left(\frac{\alpha}{2} + ir\right)\right|,$$

uniformly in $1 < \alpha \leq 2$, say, for all $r > 0$. Such an improvement would certainly narrow the gap between the upper and lower bounds for $\|f\|$. It would be of interest to find a more precise relationship between $\|f\|$ and $r$ for $c_f(1) = 1$. For example, do there exist constants $\gamma$, $\delta$ such that $\|f\| \sim \delta|\Gamma(\gamma + ir)|$ for $r \to \infty$?

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