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A REMARK ON AN EXAMPLE OF R. A. JOHNSON

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ABSTRACT. In [3] Johnson constructs an example of a second-order linear differential equation with almost periodic coefficients and with an almost automorphic behavior which we describe as Property J. In this paper we give a necessary and sufficient condition that a second order linear differential equation has Property J.

In [3] Johnson gives an example of a linear differential equation $x' = A(t)x$ where $x \in R^2$,

$$A(t) = \begin{pmatrix} a(t) & b(t) \\ 0 & -a(t) \end{pmatrix} \tag{1}$$

for suitable almost periodic functions $a(t)$ and $b(t)$, and such that the induced flow in the projective bundle $PS \times H(A)$, where $H(A)$ is the hull of A, has precisely two minimal sets M_1 and M_2. Moreover, one set M_1 is an almost periodic minimal set and the other set M_2 is an almost automorphic extension of $H(A)$ that is not almost periodic. The existence of second-order linear differential equations with almost periodic coefficients and with this almost automorphic behavior in $PS \times H(A)$ was predicted in [2]. Since this phenomenon is important for the classification of such equations, we make the following definition:

A linear differential equation $x' = A(t)x$, $x \in R^2$, is said to have *Property* J if $A(t)$ has almost periodic coefficients and there are two minimal sets M_1 and M_2 in the induced projective flow on $PS \times H(A)$ where M_1 is an almost periodic minimal set and M_2 is an almost automorphic extension of $H(A)$ that is not almost periodic.

The purpose of this note is to derive a necessary and sufficient condition for

$$x' = A(t)x, \qquad x \in R^2, \tag{2}$$

to have Property J. Before stating our result recall that the mean value of any almost periodic function $a(t)$ is given by

$$M(a) = \lim_{T\to\infty} \frac{1}{T}\int_0^T a(t)\,dt.$$

THEOREM. *A necessary and sufficient condition for equation* (2) *to have Property* J *is that there is an almost periodic Lyapunov-Perron transformation* $x = P(t)y$ (*i.e.* P, P^{-1} *and* $\dot{P}$ *are almost periodic in* t) *such that* $B = P^{-1}(AP - \dot{P})$ *is upper*

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triangular and almost periodic in t, say

$$B(t) = \begin{pmatrix} u(t) & v(t) \\ 0 & w(t) \end{pmatrix} \tag{3}$$

and the associated inhomogeneous equation

$$\xi' = (u - w)\xi + v \tag{4}$$

has a bounded solution that is not almost periodic in t. In this case, the following properties are valid:

(i) *For some functions* (u^*, v^*, w^*) *in the hull* $H(u, v, w)$ *the equation* $\xi' = (u^* - w^*)\xi + v^*$ *has a bounded almost automorphic solution that is not almost periodic.*

(ii) $M(u) = M(w) = \frac{1}{2}M(\operatorname{tr} A)$.

(iii) *The integral* $\int_0^t [u(s) - w(s)]\, ds$ *is unbounded in t.*

(iv) *At least one of the integrals* $\int_0^t [u(s) - M(u)]\, ds$, $\int_0^t [w(s) - M(w)]\, ds$ *is unbounded in t.*

(v) $v(t) \not\equiv 0$.

Proof. In order to prove this theorem we shall use the fact that if (2) has Property J and if $x = P(t)y$ is any almost periodic Lyapunov-Perron transformation then

$$y' = B(t)y \tag{5}$$

has Property J where $B = P^{-1}(AP - \dot{P})$, cf. [2].

Now assume that (2) has Property J. Then the almost periodic minimal set M_1 must be an N-fold cover of $H(A)$. (In fact, it is a 1-fold cover of $H(A)$.) The almost periodic Lyapunov-Perron transformation $x = P(t)y$ that reduces (2) to (5), where $B(t)$ is given by (3), is assured by [5, Theorem 9]. Next let (r, θ) denote the polar coordinates in the y-plane. Since $B(t)$ is upper triangular, the minimal set M_1 for (5) is generated by $\theta = 0$ (or π). The other minimal set M_2 must then be bounded away from 0 and π. This means that if $\theta(t)$ is the θ-coordinate of the solution of (5) that originates in M_2, then $\cot \theta(t)$ is bounded in t. However, $\xi(t) = \cot \theta(t)$ is necessarily a solution of (4). It is bounded and not almost periodic. On the other hand, if (4) has a bounded solution that is not almost periodic, then by [4, Proposition 3.8] statement (i) is valid. Also the argument used by Johnson [3] shows that (5) has Property J where $B(t)$ is given by (3).

In order to prove statement (ii) we shall use the properties of the spectrum $\Sigma(A)$ and $\Sigma(B) = \{M(u), M(w)\}$. Since [2] Property J implies that $\Sigma(A)$ consists of one point (which is necessarily $\{\frac{1}{2}M(\operatorname{tr} A)\}$) statement (ii) now follows. Since (4) has a bounded solution that is not almost periodic and since $M(u - w) = 0$, it follows from Favard's Theorem [1, pp. 101, 107] that statement (iii) is valid. Statement (iv) now follows immediately from (ii) and (iii). If $v(t) \equiv 0$, then it follows that $\theta = \pi/2$ generates an almost periodic minimal set M_3 in the induced projective flow on $PS \times H(A)$. In other words there are three distinct minimal sets $\{M_1, M_2, M_3\}$ in this flow. Consequently by [6, Theorem 8] the induced flow on $PS \times H(A)$ is distal. Since the restriction of this flow to M_2 is not distal, we have a contradiction. Q.E.D.

REMARK. We cannot conclude, as in Johnson's example, that $u = -w$ in (3). However if $x' = A(t)x$, $x \in R^2$, is given with almost periodic coefficients, then for any almost periodic function $\alpha(t)$ the shifted equation

$$x' = (A(t) - \alpha(t)I)x \tag{6}$$

induces the same flow on $PS \times H(A)$, cf. [5, p. 29]. Furthermore, if $x = P(t)y$ transforms (2) to (5), then this will change (6) to $y' = (B(t) - \alpha(t)I)y$. Consequently if one chooses $\alpha(t) = \frac{1}{2}\operatorname{tr} A(t)$, then the upper triangular matrix $(B - \alpha I)$ has the form (1).

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