

EQUIVALENT CONDITIONS FOR DECOMPOSABLE OPERATORS

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ABSTRACT. Several new characterizations of an arbitrary decomposable operator on Banach space are given; for example, one of these is in terms of spectral conditions on an arbitrary invariant subspace, while another uses the spectral manifold $X_T(G^-)$ (rather than $X_T(F)$). From these results a short proof of Frunzà's duality theorem is derived. Finally we give sufficient conditions that the predual of a decomposable operator is of the same class.

1. Introduction. In this paper we present several new characterizations of a decomposable operator in the sense of Foiaş (see §2, Theorem 1). Condition (3) below gives an equivalence in terms of an arbitrary invariant subspace of the given operator satisfying certain spectral properties. Another condition (5) characterizes a decomposable operator among the quasidecomposable operators. A final equivalent condition (6) is similar to one given in [6], but it is stated for open rather than closed sets.

Under the restriction that the spectrum of a given operator is nowhere dense, the operator is decomposable if and only if its adjoint is decomposable. This is proved in §3.

Throughout the rest of the paper T will denote a bounded linear operator on the complex Banach space X . A spectral maximal space (for T) is a closed invariant linear manifold M (subspace) which contains every invariant subspace N of T for which $\sigma(T|N) \subset \sigma(T|M)$. Let C be the complex plane and let $\{G_1, G_2, \dots, G_n\}$ be a finite open cover of C . Then T is said to be decomposable if there are spectral maximal spaces M_1, M_2, \dots, M_n such that $X = M_1 + \dots + M_n$ and $\sigma(T|M_i) \subset G_i$ (equivalently, $\sigma(T|M_i) \subset G_i^-$) for $i = 1, 2, \dots, n$. If $n \leq 2$, T is called 2-decomposable.

The elementary properties of a decomposable operator T may be found in [4] and can be summarized as follows. It has the single-valued extension property (SVEP), which implies that for each set $F \subset C$ the spectral manifold $X_T(F) = \{x \in X: (\lambda - T)f(\lambda) = x \text{ for some analytic } f: C \setminus F \rightarrow X\}$ is defined and invariant under T . If F is closed, then $X_T(F)$ is norm-closed and hence spectral maximal such that $\sigma(T|X_T(F)) \subset F$; moreover, every spectral maximal space has this form.

All of the above remarks hold for a 2-decomposable operator, and it is clear that T is 2-decomposable if it has the SVEP and $X = X_T(G_1^-) + X_T(G_2^-)$ for every pair of open sets G_1, G_2 covering C .

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Let X' be the dual space of X , and let $A \subset X$, $B \subset X'$. Then A^\perp [resp. B^\perp] denotes the annihilator of A in X' [resp. annihilator of B in X]. We write M^- for the norm-closure of $M \subset X$ and F° for the interior of $F \subset C$. Let M be a T -invariant subspace. Then T^M will denote the operator induced on X/M by T . For $M = X_T(F)$, we write T^F for the induced operator, and T^G if $M = X_T(G)^\perp$.

Finally, let us recall that T is quasidecomposable if it has the SVEP and the manifolds $X_T(G_i^-)$ are closed and span X for each cover $\{G_i\}$ of C ($i = 1, 2, \dots, n$).

2. The main result.

THEOREM 1. *Let T be a bounded linear operator on the Banach space X , and let T' be its adjoint on the dual space X' . Then the following assertions are equivalent.*

- (1) T is decomposable.
- (2) T is 2-decomposable.
- (3) For every open set G in C there is a T -invariant subspace Y such that $\sigma(T|Y) \subset G^-$ and $\sigma(T^Y) \subset C \setminus G$.
- (4) T has the SVEP and for F closed $X_T(F)$ is closed such that $\sigma(T^F) \subset C \setminus F^\circ$.
- (5) T is quasidecomposable and for F closed $\sigma(T'|X_T(C \setminus F)^\perp) \subset F$.
- (6) T has the SVEP and for G open $X_T(G^-)$ is closed and $\sigma(T^G) \subset C \setminus G$.

The equivalence (1) \Leftrightarrow (2) was proved by Radjabalipour [9], and (2) \Leftrightarrow (4) was proved in [6]. The characterizations (3), (5) and (6) are new. Our proof of the theorem will use the following lemmas.

LEMMA 1 [7, p. 221]. *Let M and N be closed linear manifolds in X . Then $M + N$ is closed iff $M^\perp + N^\perp$ is closed and in this case $(M \cap N)^\perp = M^\perp + N^\perp$.*

LEMMA 2 [1, p. 434]. *Suppose T has the SVEP and F_1 and F_2 are disjoint closed sets. If $X_T(F_1 \cup F_2)$ is closed, then $X_T(F_i)$ ($i = 1, 2$) are closed and $X_T(F_1 \cup F_2) = X_T(F_1) \oplus X_T(F_2)$.*

The next lemma follows from Liouville's theorem.

LEMMA 3. *Suppose T and its adjoint T' both have the SVEP. If F and H are disjoint sets in C , then $X_T(F)^\perp \supset X_{T'}(H)$ and $X_T(F) \subset X_{T'}(H)^\perp$.*

LEMMA 4 [3]. *If T has the SVEP and is surjective on X , then T is injective.*

To state the final lemma required for the proof of Theorem 1, we recall the notion of an analytically invariant subspace (see [8, §2] for details). The subspace M in X is *analytically invariant* for T if for each analytic function $f: D \rightarrow X$ the relation $(\lambda - T)f(\lambda) \in M$ for $\lambda \in D$ implies that $f(\lambda) \in M$ for $\lambda \in D$. An easy argument shows that every analytically invariant subspace is invariant.

LEMMA 5. *Let T have the SVEP. If Y is a T -invariant subspace such that $\sigma(T|Y) \cap \sigma(T^Y)$ is nowhere dense in C , then Y is analytically invariant.*

PROOF. Let $f: D \rightarrow X$ be an analytic function such that D is connected and $(\lambda - T)f(\lambda) \in Y$ for $\lambda \in D$. Suppose $E = D \cap \rho(T|Y) \neq \emptyset$ and let $h(\lambda) = (\lambda - T)f(\lambda)$ and $g(\lambda) = R(\lambda; T|Y)h(\lambda)$ for $\lambda \in E$. It follows that $(\lambda - T)g(\lambda) = (\lambda - T)f(\lambda)$ for $\lambda \in E$. Since T has the SVEP, $f(\lambda) = g(\lambda)$ for $\lambda \in E$, so $f(\lambda) \in Y$ for all $\lambda \in D$ by analytic continuation. On the other hand, $D \subset \sigma(T|Y)$ implies that $G = D \cap \rho(T^Y) \neq \emptyset$. On G we obtain $(\lambda - T^Y)\hat{f}(\lambda) = 0$ where $\hat{f}(\lambda)$ denotes the coset $f(\lambda) + Y$. Hence $\hat{f} = 0$ on G , so $f(\lambda) \in Y$ for all $\lambda \in D$.

PROOF OF THEOREM 1. Our argument will consist of proving the implications (2) \Rightarrow (4) \Rightarrow (3) \Rightarrow (2) and (4) \Rightarrow (5) \Rightarrow (6) \Rightarrow (3).

(2) \Rightarrow (4). Let $\lambda \in F^\circ$. Without loss of generality suppose $\lambda = 0$. Let H be an open set such that H and F° cover C and $0 \notin H^-$, and let M and N be spectral maximal spaces such that $X = M + N$, $\sigma(T|M) \subset F^\circ$ and $\sigma(T|N) \subset H$. For x arbitrary in X , $x = y + z$ with $y \in M$, $z \in N$. Clearly $M \subset X_T(F)$, so $x' = T^F z'$, where x' denotes the coset $x + X_T(F)$; hence T^F is surjective. Now suppose that $T^F x' = 0$ for some $x \in X$. Since $Tx \in X_T(F)$ we can prove that T^F is injective if we show $x \in X_T(F)$. Let V be the span of x and $X_T(F)$. Hence V is an invariant subspace under T . It suffices to show that $V \subset X_T(F)$ and this will follow from $\sigma(T|V) \subset F$ since $X_T(F)$ is spectral maximal. Let $\mu \notin F$. Then $\mu \neq 0$, so the equalities $\mu x = (\mu - T)x + Tx = (\mu - T)(x + y)$ for some $y \in X_T(F)$ show that x has a preimage in V under $\mu - T$. It follows that $\mu - T$ is surjective on V and hence bijective on V by Lemma 4. Thus $x \in X_T(F)$ and T^F is bijective. Hence (4) is proved.

(4) \Rightarrow (3). Let G be open and put $F = G^-$. By (4), $X_T(F)$ is closed; hence $\sigma(T|X_T(F)) \subset F$. Moreover, T^F has spectrum contained in $C \setminus G$. Thus, (3) follows if we put $Y = X_T(G^-)$.

(3) \Rightarrow (2). Let $\{G_1, G_2\}$ be any open cover of C . By [8, Theorem 3.1], we need only prove that there exist analytically invariant subspaces Y_1, Y_2 such that $X = Y_1 + Y_2$ and $\sigma(T|Y_i) \subset G_i^-$. To see this put $G = G_1 \cap G_2$ and let Y be as in (3) corresponding to G . By Lemma 5 and (3), Y is analytically invariant to T , and $\sigma(T^Y) \subset C \setminus G_1 \cap G_2 = (C \setminus G_1) \cup (C \setminus G_2)$. Since the last union is a disjoint one of closed sets, we can by the Riesz-Dunford functional calculus write the quotient space X/Y as a direct sum $W_1 \oplus W_2$ of T^Y -invariant subspaces such that $\sigma(T^Y|W_i) \subset C \setminus G_i$ ($i = 1, 2$). Let $\phi: X \rightarrow X/Y$ be the natural surjection, and put $Y_i = \phi^{-1}(W_j)$, $j \neq i$ ($i, j = 1, 2$). Then Y_i is T -invariant, $X = Y_1 + Y_2$ and $\sigma(T|Y_i) \subset \sigma(T^Y|W_j) \cup \sigma(T|Y) \subset G_i^- \cap G^- \subset \bar{G}_i$. It remains to prove that Y_i is analytically invariant to T . Let $f: D \rightarrow X$ be analytic on the connected domain D such that $(\lambda - T)f(\lambda) \in Y_1$. For $\mu \in D$ fixed there is an open disc E in D with center μ and analytic functions $f_i: E \rightarrow Y_i$ ($i = 1, 2$) such that $f = f_1 + f_2$ on E (see [2, proof of Theorem 8]). In the quotient space X/Y ,

$$(\lambda - T^Y)\hat{f}_2(\lambda) = (\lambda - T^Y)[\hat{f}(\lambda) - \hat{f}_1(\lambda)] \in \phi(Y_1) \cap \phi(Y_2) = (0),$$

where $\hat{f} = \phi \circ f$. Thus by the analytic invariance of Y , $f_2(\lambda) \in Y$ for $\lambda \in E$; and it follows that $f(\lambda) \in Y_1$ for all $\lambda \in D$. This proves that Y_1 (and hence Y_2 also) is analytically invariant, and (3) is proved.

(4) \Rightarrow (5). Since (4) \Leftrightarrow (1), it is obvious that T is quasidecomposable; hence it suffices to prove that $\sigma(T'|X_T(C \setminus F)^\perp) \subset F$. Let $G = C \setminus F$. Let $\lambda \in G$ and suppose that $\lambda = 0$. Let $\{F_n\}$ be an increasing sequence of closed sets whose union is G such that $0 \in F_1^\circ$. Then $X_T(G) = X_T(\cup F_n) = \cup X_T(F_n)$, and so $X_T(G)^\perp = \cap X_T(F_n)^\perp$. Let $W = X_T(G)^\perp$ and $W_n = X_T(F_n)^\perp$. Then W_n and W are T' -invariant and $T'|W_n$ can be identified with the adjoint of the corresponding operator induced by T on $X/X_T(F_n)$. By (4), T' is bijective on each W_n , and it follows that T' is bijective on W . Hence the desired inclusion is proved.

(5) \Rightarrow (6). By definition T has the SVEP and $X_T(G^-)$ is closed for all open G . Since $T'|X_T(G)^\perp$ is the adjoint of T^G , (6) follows from the inclusion in (5).

(6) \Rightarrow (3). Let G be open and let $Y = X_T(G)^-$. Since $X_T(G^-)$ is closed by hypothesis, $\sigma(T|Y) \subset G^-$ and the inclusion $\sigma(T^G) \subset C \setminus G$ proves (3). This completes the proof of the theorem.

The theorem allows us to prove a duality result of Frunză [5], which we shall use later.

COROLLARY 1. *Let T be 2-decomposable. Then the adjoint T' is also 2-decomposable and $X'_T(F) = X_T(C \setminus F)^\perp$ for all closed F .*

PROOF. Let $\{G_1, G_2\}$ be a cover of C and let $H_i = C \setminus G_i^-$. Then the H_i^- are closed and disjoint; hence Lemma 2 implies that $X_T(H_1^-) + X_T(H_2^-)$ is direct and thus closed. It follows that $X_T(H_1)^- + X_T(H_2)^-$ is closed in X ; hence, by Lemma 1, $X_T(H_1)^\perp + X_T(H_2)^\perp$ is closed in X' and because the H_i^- are disjoint $X' = [X_T(H_1)^- \cap X_T(H_2)^-]^\perp = X_T(H_1)^\perp + X_T(H_2)^\perp$. Put $W_i = X_T(H_i)^\perp$. Then $\sigma(T'|W_i) \subset C \setminus H_i = G_i^-$ (by (5) above). It follows from [2, Theorem 8] that T' has the SVEP. Thus $W_i \subset X'_T(G_i^-)$ and the reverse inclusion results from Lemma 3. Since $X'_T(G_i^-)$ are closed and have linear sum X' , T' is 2-decomposable. The identity $X'_T(F) = X_T(C \setminus F)^\perp$ follows from (5) and Lemma 3.

3. Preduals. In this section we seek sufficient conditions under which the implication in Corollary 1 is reversible. To do this we need the following lemma. By Corollary 1 each $X'_T(F)$ is weak*-closed. The next lemma (whose proof was corrected by the referee) may be considered a generalization of this fact.

LEMMA 6. *Let S be the adjoint operator on some dual space X' . If S has the SVEP, then every spectral maximal space of S is weak*-closed.*

PROOF. Let M be a spectral maximal space of S , and let N be the linear hull of the weak*-closure of the unit ball of M . Hence N is a norm closed S -invariant subspace of X' . Fix $u \in N$. Then u is the weak*-limit of a uniformly bounded net $\{u_\alpha\}$ in M . For $\lambda \notin \sigma(S|M)$ there is a uniformly bounded net $\{v_\alpha\}$ in X' such that $(\lambda - S)v_\alpha = u_\alpha$. Now there is $v \in X'$ every weak*-neighborhood of which contains a cofinal subnet of $\{v_\alpha\}$. It follows from the weak*-continuity of S that $(\lambda - S)v = u$, and moreover, $v \in N$. Hence $\lambda - S$ is surjective on N . Because $S|N$ also has the SVEP, by Lemma 4, $\lambda - S$ is bijective on N and thus $\sigma(S|N) \subset \sigma(S|M)$ and we have $N = M$. By the Krein-Šmulian theorem M is weak*-closed.

THEOREM 2. *Let T be an operator on X with adjoint T' on X' . Let S be the adjoint of T' , and suppose that X (considered as subspace of X'') is analytically invariant under S . Then T is decomposable if T' is decomposable. In particular, if $\sigma(T)$ is nowhere dense, then T' is decomposable if and only if T is.*

PROOF. By (1) \Leftrightarrow (2) of Theorem 1 we may prove the present result in terms of 2-decomposability. Let T' be 2-decomposable and let $\{G_1, G_2\}$ be an open cover of C . Let $W_i = X'_T(C \setminus G_i)^-$ and put $M_i = W_i^\perp$ ($i = 1, 2$). Then M_i^\perp is the weak*-closure of W_i in X' . Letting $F_i = C \setminus G_i$, we see that the F_i are closed disjoint sets; hence the inclusion $\sigma(T'|W_i) \subset F_i$ implies that $M_1^\perp + M_2^\perp$ is closed by Lemmas 2 and 6. Hence $M_1 + M_2$ is closed by Lemma 1 and thus $X = M_1 + M_2$ follows from the relation $(M_1 + M_2)^\perp = M_1^\perp \cap M_2^\perp = (0)$. Since S is decomposable by Corollary 1, it has the SVEP and so the restriction $T = S|X$ has this property also. Thus the linear manifolds $X_T(G_i^-)$ are defined.

We complete the proof of the first conclusion of the theorem by showing that $M_i = X_T(G_i^-)$, i.e. $X_T(G_i^-)$ are closed and their linear sum is X . By Lemma 3 we have $X_T(G_i^-) \subset W_i^\perp = M_i$. For the reverse inclusion, note that, by the last statement of Corollary 1, $X_S''(G_i^-) = W_i^a =$ the annihilator of W_i in X'' (the dual of X') and, by Corollary 1, (4) of Theorem 1 and Lemma 5, $X_S''(G_i^-)$ is analytically invariant under S . Now let $V_i = X \cap X_S''(G_i^-)$ where X is considered a subspace of X'' . Intersections of analytically invariant subspaces are obviously analytically invariant; hence by hypothesis and the previous remark V_i is analytically invariant under S . By [8, Proposition 1.13(1), p. 231], V_i is analytically invariant under $S|X_S''(G_i^-)$; hence by [8, Corollary 1.4, p. 227], $\sigma(S|V_i) \subset \sigma(S|X_S''(G_i^-)) \subset G_i^-$. Regarding V_i now as T -invariant, we see that $M_i \subset V_i \subset X_T(G_i^-)$ and we infer that $X_T(G_i^-)$ are closed. Hence T is 2-decomposable.

To prove the final assertion of the theorem, merely note that if $\sigma(T)$ is nowhere dense, then by Lemma 5 and by [8, Proposition 1.3, p. 227], X is analytically invariant under S . Now apply the first part of the theorem and Corollary 1.

REMARK. The hypotheses of Theorem 2 make the following question natural: Is the Banach space X analytically invariant under the second adjoint T'' of an operator T (on X), at least if T' is decomposable? A positive answer would, of course, give the general result that the predual of any decomposable operator is decomposable.

Added September, 1980. Since this paper was written, the author has learned that Ș. Frunzã has proved Theorem 2 in general, i.e. the predual of a decomposable operator is decomposable. It might also be added that the implication (3) \Rightarrow (2) in the proof of Theorem 1 may be shortened by using the fact that T has the spectral decomposition property [2], and that such operators are decomposable (see E. Albrecht, *On decomposable operators*, Integral Equations Operator Theory 2 (1979), 1-10).

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