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HOMOGENEOUS TREE-LIKE CONTINUA

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ABSTRACT. We prove that every k-junctioned homogeneous tree-like continuum is chainable, and hence a pseudo-arc. Possible extensions of this result are briefly discussed.

In 1959, Bing [**B**] proved that the pseudo-arc is the only homogeneous, chainable continuum. No other homogeneous tree-like continuum is known. We prove that if M is a k-junctioned homogeneous tree-like continuum, then M is chainable (and hence a pseudo-arc).

Burgess [**Bu**] has shown that every proper subcontinuum of a homogeneous k-junctioned tree-like continuum is a pseudo-arc. Extensive use will be made of this fact. We will also use the following theorem, proven by Hagopian [**H**], which follows from a result of Effros [**E**].

THEOREM. *Let M be a homogeneous continuum, and $\varepsilon > 0$. There exists $\delta > 0$ such that if $x, y \in M$ and* $\operatorname{dist}(x, y) < \delta$, *there is a homeomorphism h of M with $h(x) = y$ and* $\operatorname{dist}(z, h(z)) < \varepsilon$ *for each $z \in M$.* $\square$

If P is a continuum and C is a chain, we will say that C *essentially covers* P provided C covers P but no proper subchain of C covers P. Other terminology (chain, pattern, amalgamation, etc.) and facts about hereditarily indecomposable continua which we use are standard. As usual, if H is a collection of sets then H^* is the union of the elements of H.

We will now proceed directly to the proof of the main theorem.

THEOREM. *Every k-junctioned homogeneous tree-like continuum M is chainable.*

PROOF. Let U be a tree covering of mesh less than ε covering M. Let δ be a Lebesgue number for U which is also smaller than the distance between nonintersecting links of U. Choose $\gamma > 0$ such that, if $x, y \in M$ and $\operatorname{dist}(x, y) < \gamma$, there exists a homeomorphism h of M with $h(x) = y$ and $\operatorname{dist}(z, h(z)) < \delta/15k$ for each $z \in M$.

Let V be a k-junctioned tree chain of mesh less than γ which refines U and covers M. Let A be the collection of chains in V which are maximal with respect to

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containing no junction link of V as an interior link. For each $\alpha \in A$, let P_α be a pseudo-arc in M which is essentially covered by α and intersects no links of V not in α.

Let P_α^0 and P_α^1 be points of P_α, in different composants, in the opposite end links of α. If $\alpha \cap \alpha' \neq \varnothing$, and P_α^i, $P_{\alpha'}^j$ are in the common link of α and α', then there is a homeomorphism $h_{(\alpha,\alpha')}$ of M, moving no point more than $\delta/15k$, with $h_{(\alpha,\alpha')}(P_\alpha^i) = P_{\alpha'}^j$. By the hereditary indecomposability of M, $h_{(\alpha,\alpha')}(P_\alpha) \subset P_{\alpha'}$ or $h_{(\alpha,\alpha')}(P_\alpha) \supset P_{\alpha'}$.

By composing the $h_{(\alpha,\alpha')}$'s (or their inverses) we can obtain $\tilde{\alpha} \in A$ and homeomorphisms h_α, each moving no point more than $2\delta/15$, such that $h_\alpha(P_\alpha) \subset P_{\tilde{\alpha}}$ for each $\alpha \in A$.

We shall use these homeomorphisms and the pattern followed by $\tilde{\alpha}$ in U to construct an ε-chain covering M (and refining U).

Let $A_0 = \{\tilde{\alpha}\}$ and, for each $i \in \omega_0$, let $A_{i+1} = \{\alpha \in A | \alpha \cap (A_i^*) \neq \varnothing$, but $\alpha \notin A_i\}$. Let g be a pattern which $\tilde{\alpha}$ follows in U, chosen such that if $g(\alpha) = \beta$ then the $\delta/3$-neighborhood of α is contained in β.

We will modify the chains α slightly before doing any amalgamation. If $\alpha \in A_i$ and a is an end link of α such that every other chain $\alpha' \in A$ containing a satisfies $\alpha' \in A_{i+1}$, then split a into links, one, L_α, for α and one, $L_{\alpha'}$, for each other α' containing a, such that $P_\alpha \cap L_{\alpha'} = \varnothing$ for each α', $L_\alpha \cap P_{\alpha'} = \varnothing$ for each α', and $L_{\alpha'} \cap L_{\alpha''} = \varnothing$ for each distinct $\alpha', \alpha'' \in A_{i+1}$ containing a.

We will now amalgamate modifications of these altered chains α into a single chain W, of mesh less than ε, which follows the pattern g in U. For each α, let C_α be a chain covering P_α and refining α such that the image of each link of C_α under the homeomorphism h_α is a subset of a link of $\tilde{\alpha}$. We can choose C_α such that its boundary is contained only in its end links, which are in the end links of α and contain P_α^0 and P_α^1 respectively. Let g_α be a pattern, respecting end links, which C_α follows in α. By using the fact that every proper subcontinuum of M is a pseudo-arc, we can use the same type argument as used in the proof of Theorem 3 of [L] to show that the part of M in the modified α can be amalgamated into a chain D_α such that (1) D_α follows the pattern g_α in α, (2) for each n, the nth link of C_α is in the nth link of D_α, and (3) the part of M in the intersection of the link of the modified α containing P_α^i $(i = 0, 1)$ with links not in the modified α is amalgamated into the same link of D_α as the point P_α^i.

The desired chain W can now be constructed. For each $\alpha \in A - \{\tilde{\alpha}\}$ and positive integer n, choose a link $L_{\alpha,n}$ of $\tilde{\alpha}$ which contains the image under the homeomorphism h_α of the nth link of C_α. If b is a link of $\tilde{\alpha}$, the corresponding link of W is b together with the link of D_α containing the nth link of C_α where $b = L_{\alpha,n}$, for each α and each n.

Our choices of g and of the h_α's guarantee that each link of W is within $\delta/3$ of the link of $\tilde{\alpha}$ it contains. The correspondence between D_α and C_α guarantees that each amalgamation of D_α is a chain, and our choice of h_α's (obtained by composing $h_{\alpha,\alpha'}$'s), modification of the end links of the α's, and construction of the D_α's guarantee that D_α and $D_{\alpha'}$ together form a chain when amalgamated into W. $\square$

The fact that M was k-junctioned allowed us to form the h_α's such that none of them moved any point more than $2\delta/15$. Being k-junctioned also implied that M

was hereditarily indecomposable with every subcontinuum a pseudo-arc–which was crucial to our argument.

It is conceivable that a hereditarily indecomposable, non-k-junctioned, homogeneous tree-like continuum exists (perhaps a variation on Ingram's [I] examples). Without chainable subcontinua, our techniques give one little to work with, even if one knows the continuum is hereditarily indecomposable.

Actually in the non-k-junctioned case, one does not know beforehand whether a homogeneous tree-like continuum must be hereditarily indecomposable. In fact it is still unknown whether a homogeneous tree-like continuum can contain an arc. It is known by a result of Jones [J] that a homogeneous tree-like continuum must be indecomposable. Hagopian [H2] and Jones [J2] have shown that every homogeneous tree-like plane continuum is hereditarily indecomposable.

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