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SPACES FOR WHICH ALL COMPACT METRIC SPACES ARE REMAINDERS

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ABSTRACT. Let X be a locally compact, completely regular, Hausdorff space, and let $K(X)$ be the lattice of compactifications of X. Conditions on $K(X)$ and an internal condition are obtained which characterize when X has all compact metric spaces as remainders.

1. Introduction. Throughout this paper X will denote a noncompact, locally compact, completely regular, Hausdorff space. A remainder of X is any $\alpha X - X$, where αX is a Hausdorff compactification of X. One of the major concerns in the study of remainders has been the problem of characterizing when all members of a certain class of spaces can serve as remainders for each X in another class of spaces (cf. **[1]**, **[2]**, **[3]**, **[7]**, **[8]**, **[11]**, **[12]**, etc.). Let $K(X)$ denote the complete lattice of compoactifications of X (see **[6]**). The purpose of this paper is to determine those spaces for which all compact metric spaces are remainders. (Clearly, such spaces must be locally compact.) An internal characterization and a characterization in terms of $K(X)$ are obtained.

2. Characterization. In general, notation and terminology concerning remainders will follow that of **[2]**. For convenience, we shall say X has a countable remainder whenever some $\alpha X - X$ is countably infinite. For αX, $\gamma X \in K(X)$, we recall that $\alpha X \geqslant \gamma X$ if and only if there exists a continuous mapping of αX onto γX which is the identity on X. Let βX denote the Stone-Čech compactification of X and let N denote the natural numbers.

THEOREM 2.1. *For locally compact X, the following are equivalent*:

(A) *There exists a chain $\{\alpha_n X | n \in N\}$ in $K(X)$, where $\alpha_n X - X = \{a_i^n | i = 1, \dots, 2^n\}$ and where $\alpha_{n+1}X \geqslant \alpha_n X$ under mappings t_{n+1} which satisfy $t_{n+1}(a_{2i-1}^{n+1}) = t_{n+1}(a_{2i}^{n+1}) = a_i^n$, for $i = 1, \dots, 2^n$.*

(B) *Every compact metric space is a remainder of X.*

(C) *There exists a sequence of families $\mathcal{G}_n$ of pairwise disjoint, nonempty, open subsets of X such that for each $n \in N$, $\mathcal{G}_n = \{G_i^n | i = 1, \dots, 2^n\}$ and*

(i) $G_{2i-1}^{n+1} \cup G_{2i}^{n+1} \subseteq G_i^n$, $i = 1, \dots, 2^n$,

(ii) $K_n = X - \bigcup \{G_i^n | i = 1, \dots, 2^n\}$ *is compact,*

(iii) $K_n \cup G_i^n$ *is noncompact for each* $i = 1, \dots, 2^n$.

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PROOF. (A) implies (B). Take $\{\alpha_n X | n \in N\}$ as in (A). For each $n \in N$, let f_n be the natural mapping of $\beta X - X$ onto $\alpha_n X - X$ and where f_n is the identity on X (cf. [4]). Set $U_i^n = f_n^{-1}(a_i^n)$, $i = 1, \ldots, 2^n$, $V_n = \cup \{U_i^n | 1 < i < 2^n; i \text{ odd}\}$ and $W_n = \cup \{U_i^n | 1 < i < 2^n; i \text{ even}\}$, for each $n \in N$. Then V_n and W_n partition $\beta X - X$ into disjoint, nonempty open sets. Let $X_n = X \cup \{b_n, c_n\}$ be the two-point compactification of X determined by identifying V_n and W_n to points b_n and c_n, respectively. Let p_n be the natural mapping of βX onto X_n, for each $n \in N$. Take $Y = \times_{n \in N} X_n$ and denote points of Y by (y_n), where $y_n \in X_n$. Embed X in Y by letting $\varphi(x) = (y_n)$, where $y_n = x$, for all $n \in N$. Suppose $(d_n) \in Y$ satisfies $d_n = b_n$ or $d_n = c_n$, for each $n \in N$. Our aim is to show that $\mathrm{Cl}_Y\, \varphi(X) - \varphi(X)$ consists of precisely such points. To this end, let π_n be the projection of Y onto X_n and suppose that $G = \pi_{n_1}^{-1}(G_{n_1}) \cap \cdots \cap \pi_{n_k}^{-1}(G_{n_k})$ is a basic neighborhood of (d_n) in Y. If $d_{n_i} = b_{n_i}$, then $p_{n_i}^{-1}(G_{n_i})$ contains V_{n_i} and if $d_{n_i} = c_{n_i}$, then $W_{n_i} \subseteq p_{n_i}^{-1}(G_{n_i})$. Now, for $n > 1$, $t_{n+1} \circ f_{n+1} = f_n$, so that for $i = 1, 2, \ldots, 2^{n+1}$, $U_i^{n+1} \subseteq U^n_{[(i+1)/2]}$, where [ ] is the greatest integer function. It follows that some $U_j^{n_k} \subseteq p_{n_i}^{-1}(G_{n_i})$, for all $i = 1, \ldots, k$. Since $U_j^{n_k}$ is nonempty and X is dense in βX, we can select $w \in X$ such that $w \in p_{n_1}^{-1}(G_{n_1}) \cap \cdots \cap p_{n_k}^{-1}(G_{n_k})$. Then $\varphi(w) \in G \cap \varphi(X)$, so that $(d_n) \in \mathrm{Cl}_Y\, \varphi(X) - \varphi(X)$.

Next, suppose $(d_n) \in Y$, where each $d_n \in X$ but $d_i \neq d_j$, for some $i \neq j$. Then there exist disjoint neighborhoods U and V in X of d_i and d_j, respectively. Now $\pi_i^{-1}(U) \cap \pi_j^{-1}(V)$ is a neighborhood of (d_n) which contains no point of $\varphi(X)$. Similarly, if, for some $i \neq j$, $d_i \in X$ and $d_j \in X_j - X$, then $(d_n) \notin \mathrm{Cl}_Y\, \varphi(X) - \varphi(X)$. Thus, $\mathrm{CL}_Y\, \varphi(X)$ is a compactification of X whose remainder is a homeomorph of the (usual) Cantor set $\mathcal{C}$. It follows from a theorem of Magill [9] that any compact metric space is a remainder of X.

(B) *implies* (C). Let αX be a compactification of X with remainder $\mathcal{C}$. For each $n \in N$, let $\{A_i^n | i = 1, \ldots, 2^n\}$ be a collection of closed subsets of $\mathcal{C}$ such that $\mathcal{C} = \cup \{A_i^n | i = 1, \ldots, 2^n\}$ and $A_{2i-1}^{n+1} \cup A_{2i}^{n+1} = A_i^n$, $i = 1, \ldots, 2^n$. In αX choose disjoint open sets H_i^1 with $A_i^1 \subseteq H_i^1$, for $i = 1, 2$. Set $G_i^1 = H_i^1 \cap X$, $i = 1, 2$. Clearly $K_1 = X - (G_1^1 \cup G_2^1)$ is compact but $K_1 \cup G_1^1$ and $K_1 \cup G_2^1$ are noncompact. Proceeding inductively, assume that a collection $\mathcal{G}_n$ has been defined as in (C). Select a pairwise disjoint family $\{H_i^{n+1} | i = 1, \ldots, 2^{n+1}\}$ of open sets in αX, such that $A_i^{n+1} \subseteq H_i^{n+1}$, $i = 1, \ldots, 2^{n+1}$, and set $G_{2i-1}^{n+1} = H_{2i-1}^{n+1} \cap G_i^n$ and $G_{2i}^{n+1} = H_{2i}^{n+1} \cap G_i^n$. This defines $\mathcal{G}_{n+1}$ according to (C). Hence the sequence $\{\mathcal{G}_n | n \in N\}$ satisfies (C).

(C) *implies* (A). We utilize Magill's construction in [7] to obtain a sequence of compactifications $\{\alpha_n X | n \in N\}$ subject to (A). Accordingly, let $\alpha_n X - X = \{a_i^n | i = 1, \ldots, 2^n\}$, where basic neighborhoods of each a_i^n are sets $\mathcal{O} \cup \{a_i^n\}$, for $\mathcal{O}$ open in X and $(K_n \cup G_i^n) - \mathcal{O}$ compact.

Define mappings t_{n+1} from $\alpha_{n+1} X$ onto $\alpha_n X$ as in (A). Evidently, each t_{n+1} is continuous at points of X since X is locally compact. Next, let $\mathcal{O} \cup \{a_i^n\}$ be any basic neighborhood of a_i^n. Consider $\mathcal{O} \cup \{a_{2i}^{n+1}\}$. Since $[(K_{n+1} \cup G_{2i}^{n+1}) - \mathcal{O}] \subseteq [K_{n+1} - \mathcal{O}] \cup [(K_n \cup G_i^n) - \mathcal{O}]$, $(K_{n+1} \cup G_{2i}^{n+1}) - \mathcal{O}$ is a closed subset of a compact set. Thus $(K_{n+1} \cup G_{2i}^{n+1}) - \mathcal{O}$ is compact and $\mathcal{O} \cup \{a_{2i}^{n+1}\}$ is a neighborhood of a_{2i}^{n+1} in $\alpha_{n+1}X$. It now follows that t_{n+1} is continuous at a_{2i}^{n+1}. In this manner t_{n+1} is demonstrated to be continuous at each point of $\alpha_{n+1}X$ and the sequence $\{\alpha_n X | n \in N\}$ satisfies (A). This complete the proof.

3. Sufficiency conditions and examples. The following is immediate from Theorem 2.1.

COROLLARY 3.1. (A) *If X contains a family $\{G_n | n \in N\}$ of pairwise disjoint open sets such that $K = X - \bigcup \{G_n | n \in N\}$ is compact and all $K \cup G_n$ are noncompact, then all compact metric spaces are remainders of X.*

(B) *If X is the (topological) free union of a compact space and an infinite discrete space, then all compact metric spaces are remainders of X.*

The converse of 3.1(A) is false. For, if X is the closed unit square with $\mathcal{C} \times \{0\}$ deleted, then all compact metric spaces are remainders of X, but X contains no family of open sets satisfying the requirements of 3.1(A).

Evidently, if X satisfies (A)–(C) of Theorem 2.1, then X has a countable remainder. The converse is false. If $X = W \times W^*$, where W is the space of all countable ordinals, then $\beta X - X = W^*$ (cf. 8L and 8M of [**4**]). Since any compact metric space which is a continuous image of W^* must be countable or finite, it follows from Magill's theorem [**9**] that (B) of 2.1 cannot hold for X. However, X has a compactification with countable remainder since $\beta X - X$ has infinitely many components (cf. [**8**]).

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