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PERIODS OF PERIODIC POINTS OF MAPS OF THE CIRCLE WHICH HAVE A FIXED POINT

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ABSTRACT. For a continuous map f of the circle to itself, let $P(f)$ denote the set of positive integers n such that f has a periodic point of (least) period n. Results are obtained which specify those sets, which occur as $P(f)$, for some continuous map f of the circle to itself having a fixed point. These results extend a theorem of Šarkovskii on maps of the interval to maps of the circle which have a fixed point.

1. Introduction. This paper extends the theorem of Šarkovskii on maps of the interval to maps of the circle which have a fixed point.

Let R denote the real line, I a closed bounded interval on R, and S^1 the circle. Let $C^0(X, Y)$ denote the set of continuous maps from X to Y. For $f \in C^0(I, R)$ or $f \in C^0(S^1, S^1)$ let $P(f)$ denote the set of positive integers n such that f has a periodic point of (least) period n.

Let N denote the set of positive integers and let Δ denote the ordering of N:

$$3 \Delta 5 \Delta 7 \Delta \cdots \Delta 2 \cdot 3 \Delta 2 \cdot 5 \Delta \cdots \Delta 2^2 \cdot 3 \Delta 2^2 \cdot 5 \Delta \cdots \Delta 2^3 \Delta 2^2 \Delta 2 \Delta 1.$$

The following theorem is proved in [2], [3] and [4].

THEOREM (ŠARKOVSKII). *Let $f \in C^0(I, R)$. If $n \in P(f)$ and $n \Delta k$ then $k \in P(f)$. Conversely, suppose $S \subset N$ with the property that if $n \in S$ and $n \Delta k$ then $k \in S$. Then there is a map $f \in C^0(I, I)$ with $P(f) = S$.*

Note that the theorem of Šarkovskii completely specifies those subsets of N which occur as $P(f)$ for some $f \in C^0(I, R)$. In this paper we do the same for $f \in C^0(S^1, S^1)$ having a fixed point. Let Δ denote the ordering defined above, and let $<$ denote the usual ordering of N. The main result of this paper is the following.

THEOREM A. *Let $f \in C^0(S^1, S^1)$. Suppose $1 \in P(f)$ and $n \in P(f)$ for some integer $n > 1$. Then (at least) one of the following holds.*

(i) *For every integer m with $n < m$, $m \in P(f)$.*

(ii) *For every integer m with $n \Delta m$, $m \in P(f)$.*

We remark that in [2] the periodic points and topological entropy of maps $f \in C^0(S^1, S^1)$ are studied by examining separately the four cases where the degree of f is 0, 1, -1, or of absolute value greater than 1. The results of [2] imply that Theorem A holds in all cases except where the degree of f is -1 and n is even. The

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proof of Theorem A given here treats maps of all degrees simultaneously (including the case left open in [2]), using ideas from [1] and [2].

Let $f \in C^0(S^1, S^1)$ and suppose f has degree -1. One of the results of [2] states that if $n \in P(f)$ and n is odd then statement (ii) (in Theorem A) must hold. Now, suppose $n \in P(f)$ and n is even. By Theorem A, either (i) or (ii) holds. Suppose (i) holds. Then $(n + 1) \in P(f)$. Since $n + 1$ is odd, the result of [2] implies that $m \in P(f)$ for every positive integer m with $(n + 1) \Delta m$. Since $(n + 1) \Delta n$, (ii) holds. Hence, we have the following.

COROLLARY B. *Let $f \in C^0(S^1, S^1)$ and suppose f has degree -1. If $n \in P(f)$ then $m \in P(f)$ for every integer m with $n \Delta m$.*

The final result of this paper is the following.

THEOREM C. *Let $S \subset N$ with $1 \in S$. Suppose that for every $n \in S$ with $n > 1$ (at least) one of the following holds.*

(i) *For every integer m with $n < m$, $m \in S$.*

(ii) *For every integer m with $n \Delta m$, $m \in S$. Then there is a map $f \in C^0(S^1, S^1)$ such that $P(f) = S$.*

The proof of Theorem C is obtained by using an example from [1] for $f \in C^0(S^1, S^1)$ with $P(f) = \{1\} \cup \{k \in N\colon\ k > n\}$. This example is modified to include an invariant interval on S^1 with periodic points as specified by the theorem of Šarkovskii. Note that the example constructed has degree one. It follows from Corollary B and the results of [2] that this is the only possible degree.

2. Preliminary definitions and results. Let $f \in C^0(S^1, S^1)$. Let f^0 denote the identity map of S^1, and for any $n \in N$ define f^n inductively by $f^n = f \circ f^{n-1}$.

Let $x \in S^1$. We say x is a fixed point of f if $f(x) = x$. If x is a fixed point of f^n, for some $n \in N$, we say x is a periodic point of f. In this case the smallest element of $\{n \in N: f^n(x) = x\}$ is called the period of x.

We define the orbit of x to be $\{f^n(x): n = 0, 1, 2, \dots\}$. If x is a periodic point of f of period n, we say the orbit of x is a periodic orbit of period n. In this case the orbit of x contains exactly n points, each of which is a periodic point of period n.

We will use the following notation throughout this paper.

Notation. Let $a \in S^1$ and $b \in S^1$ with $a \neq b$. We write $[a, b]$, (a, b), $(a, b]$, or $[a, b)$ to denote the closed, open, or half-open interval from a counterclockwise to b.

We will also use the following definition.

DEFINITION. *Let I and J be proper closed intervals on S^1 and let $f \in C^0(S^1, S^1)$. We say I f-covers J if, for some closed interval $K \subset I$, $f(K) = J$.*

We conclude this section by stating three lemmas from [1] which will be used in the next section.

LEMMA 1 (LEMMA 1 OF [1]). *Let $I = [a, b]$ be a proper closed interval on S^1 and let $f \in C^0(S^1, S^1)$. Suppose $f(a) = c$ and $f(b) = d$ and $c \neq d$. Then either I f-covers $[c, d]$ or I f-covers $[d, c]$.*

LEMMA 2 (LEMMA 2 OF [1]). *Let I and J be proper closed intervals on S^1 such that I f-covers J. Suppose L is a closed interval with $L \subset J$. Then I f-covers L.*

LEMMA 3 (LEMMA 7 OF [1]). *Let $f \in C^0(S^1, S^1)$ and let P be a periodic orbit of period m where $m \geqslant 3$. Suppose that $\{M_1, \ldots, M_k\}$ is a collection of closed intervals with $2 \leqslant k \leqslant m$ such that*

(1) *for each $j \in \{1, \ldots, k\}$, there are no elements of P in the interior of M_j.*

(2) *If $i \neq j$, M_i and M_j have disjoint interiors.*

(3) *If $j \in \{2, \ldots, k\}$ the endpoints of M_j are in P.*

(4) *If b is an endpoint of M_1 then either $b \in P$ or b is a fixed point of f.*

(5) *For each $j \in \{1, \ldots, k - 1\}$, M_j f-covers M_{j+1}.*

(6) *M_1 f-covers M_1 and M_k f-covers M_1. Then for any positive integer $n > k$, $n \in P(f)$.*

3. Proof of Theorem A.

CONVENTION. *In Theorems* A_1 *and* A_2 *in this section, we assume that $f \in C^0(S^1, S^1)$ and f has a fixed point e. Also, we suppose that f has a periodic orbit $P = \{p_1, \ldots, p_n\}$ of period $n \geqslant 3$ where $P \cap (p_k, p_{k+1}) = \varnothing$ for $k = 1, \ldots, n - 1$ and $P \cap (p_n, p_1) = \varnothing$. Finally, we let $I_1 = [p_1, p_2]$, $I_2 = [p_2, p_3], \ldots, I_{n-1} = [p_{n-1}, p_n]$, and $I_n = [p_n, p_1]$, and set $A = \{I_1, \ldots, I_n\}$.*

THEOREM A_1. *Suppose that for each $I_j \in A$ there is some $I_k \in A$ with $k \neq j$ such that I_k f-covers I_j. Then for every integer m with $n < m$, $m \in P(f)$.*

PROOF. Since the fixed point e of f must be in one of the intervals $I_j \in A$, we may assume without loss of generality that $e \in I_n$. We have two cases.

Case 1. *I_n f-covers I_n.*

Let $K_1 = I_n$ and let $A_1 = \{I_j \in A\colon K_1\, f\text{-covers } I_j\}$. It follows from Lemmas 1 and 2 and the fact that P is a periodic orbit that A_1 contains at least one element I_j of A with $j \neq n$. Also, since I_n f-covers I_n, $I_n \in A_1$.

Suppose that $A_1 \neq A$. Let K_2 denote the union of the intervals in A_1. It follows from Lemmas 1 and 2 that K_2 is connected. Hence K_2 is a proper closed interval on S^1. Let $A_2 = \{I_j \in A\colon K_2\, f\text{-covers } I_j\}$. Then $A_1 \subset A_2$.

We will show that $A_2 \neq A_1$. First suppose there are at least two distinct elements of A not in A_1. Then there is an element of P not in K_2. Since P is a periodic orbit, it follows (using Lemmas 1 and 2) that $A_2 \neq A_1$. Now suppose there is exactly one element I_s of A not in A_1. By hypothesis, for some $I_t \in A$ with $t \neq s$, I_t f-covers I_s. Since $I_t \neq I_s$, $I_t \in A_1$. Hence $I_s \in A_2$, so $A_2 \neq A_1$.

Now, let $A_i = \{I_j \in A\colon K_i\ f\text{-covers } I_j\}$ and let K_{i+1} denote the union of the intervals in A_i. Then as above it follows that if $A_i \neq A$ then A_i is a proper subset of A_{i+1}. Thus, for some positive integer r with $r < n$, $A_r = A$.

We claim that for any positive integer i with $2 \leqslant i \leqslant r$, if $I_j \in A_i$ then I_u f-covers I_j for some $I_u \in A_{i-1}$. To see this, suppose that $I_j \in A_i$. Then since K_i f-covers I_j, $f(D) = I_j$ for some closed interval $D \subset K_i$. There is a closed interval $E \subset D$ such that $f(E) = I_j$ and f maps the interior of E to the interior of I_j. Hence, there are no elements of P in the interior of E. Thus, $E \subset I_u$ for some $I_u \in A_{i-1}$, and the claim is established.

Now, since $A_r = A$, our hypothesis implies that some element of A_r other than I_n f-covers I_n. Let w denote the smallest positive integer such that some element of A_w, other than I_n, f-covers I_n. Let L_1 denote an element of A_w such that $L_1 \neq I_n$ and

L_1 f-covers I_n. If $w > 1$, let L_2 denote an element of A_{w-1} such that L_2 f-covers L_1. Continuing we obtain distinct elements of A, $L_1, L_2, \ldots, L_w$ with $L_i \in A_{w+1-i}$ for $i = 1, \ldots, w$ such that L_1 f-covers I_n and L_i f-covers L_{i-1} for $i = 2, \ldots, w$. Let $k = w + 1$, and let $M_1 = I_n$, $M_2 = L_w$, $M_3 = L_{w-1}, \ldots, M_k = L_1$. Then $k \leqslant n$ and $\{M_1, \ldots, M_k\}$ is a collection of closed intervals satisfying the hypothesis of Lemma 3. Hence, by Lemma 3, $m \in P(f)$ for every integer $m > n$.

Case 2. I_n does not f-cover I_n.

By continuity, $\exists x \in [e, p_1]$ such that $f(x) \in \{p_1, p_n\}$. Hence, $\exists a \in [e, p_1]$ such that $f(a) \in \{p_1, p_n\}$ and, for all $x \in (e, a)$, $f(x) \notin \{p_1, p_n\}$. Similarly, $\exists b \in [p_n, e]$ such that $f(b) \in \{p_1, p_n\}$ and, for all $x \in (b, e)$, $f(x) \notin \{p_1, p_n\}$.

Suppose that $f(a) = p_n$ and $f(b) = p_1$. Then $f([b, a]) = [p_n, p_1]$. This is a contradiction since $I_n = [p_n, p_1]$ does not f-cover itself. Hence either $f(a) = p_1$ or $f(b) = p_n$. Without loss of generality we may assume that $f(a) = p_1$. Then (by Lemma 2) $[e, p_1]$ f-covers $[e, p_1]$.

Suppose that $f(x) = p_n$ for some $x \in [e, p_1]$. By choice of a, $x \in (a, p_1]$. By Lemma 1, the interval $[a, x]$ f-covers either $I_n = [p_n, p_1]$ or $[p_1, p_n]$. Since I_n does not f-cover itself, $[a, x]$ f-covers $[p_1, p_n]$. Thus, $[e, p_1]$ f-covers $[p_1, p_n]$. By Lemma 2, $[e, p_1]$ f-covers I_j for every $I_j \in A$ with $j \neq n$. By hypothesis I_s f-covers $[e, p_1]$ for some $I_s \in A$ with $s \neq n$. Hence, the conclusion of this theorem follows from Lemma 3 (with $k = 2$, $M_1 = [e, p_1]$, and $M_2 = I_s$). Thus, we may assume that $f(x) \neq p_n$ for all $x \in [e, p_1]$.

Now, we modify the argument of Case 1, replacing $A = \{I_1, \ldots, I_n\}$ by $A_0 = \{[e, p_1], I_1, \ldots, I_{n-1}\}$, and starting with $K_1 = [e, p_1]$ instead of $K_1 = I_n$. We let $A_i = \{I \in A_0\colon K_i\ f\text{-covers } I\}$ and let K_{i+1} denote the union of the intervals in A_i. It follows from the previous paragraph that $A_1 = \{[e, p_1], I_1, \ldots, I_t\}$ for some positive integer t.

By hypothesis, some element of $\{I_1, \ldots, I_{n-1}\}$ f-covers I_n. Also, since P is a periodic orbit, if A_i does not contain an interval which f-covers I_n then A_i is a proper subset of A_{i+1}. Hence, for some positive integer r with $1 \leqslant r \leqslant n - 1$, A_r contains an interval $I_s \in \{I_1, \ldots, I_{n-1}\}$ such that I_s f-covers I_n. By Lemma 2, I_s f-covers $[e, p_1]$. As in Case 1, we obtain a collection of closed intervals $\{M_1, \ldots, M_k\}$ (here $M_1 = [e, p_1]$) with $2 \leqslant k \leqslant n$, satisfying the hypothesis of Lemma 3. Hence, the conclusion of this theorem follows from Lemma 3. Q.E.D.

THEOREM A_2. *Suppose that for some $I_j \in A$ there does not exist $I_k \in A$ with $k \neq j$ such that I_k f-covers I_j. Then for every positive integer m with $n \Delta m$, $m \in P(f)$.*

PROOF. Let I_j be as in the hypothesis and let K denote the closure of the complement of I_j in S^1. Let $h\colon K \to I$ be a homeomorphism from K onto a closed interval I on the real line.

Our hypothesis implies that there is a continuous map $g\colon I \to R$ such that, for all $x \in K$, $f(x) \in K$ if and only if $g(h(x)) \in I$ and in this case $h(f(x)) = g(h(x))$. Thus, since the restriction of f to K has a periodic orbit of period n, $n \in P(g)$. By the theorem of Šarkovskii, $m \in P(g)$ for every positive integer m with $n \Delta m$. Hence, $m \in P(f)$ for every positive integer m with $n \Delta m$. Q.E.D.

Theorem A follows immediately from Theorems A_1 and A_2.

4. Proof of Theorem C.

LEMMA 4. *Let $I = [a, b]$ be an interval on the real line, and let k be a positive integer. Let $S = \{k\} \cup \{j \in N\colon k \,\Delta\, j\}$. There is a map $g \in C^0(I, I)$ such that $g(a) = a$, $g(b) = b$, and $P(g) = S$.*

PROOF. Let c and d be points in I with $a < c < d < b$. By the theorem of Šarkovskii, there is a continuous map $g_0\colon [c, d] \to [c, d]$ such that $P(g_0) = S$. There is a unique $g \in C^0(I, I)$ such that $g(a) = a$, $g(b) = b$, $g(x) = g_0(x)$ for all $x \in [c, d]$, and g is linear on each of the intervals $[a, c]$, $[d, b]$. Clearly $P(g) = P(g_0) = S$. Q.E.D.

THEOREM C. *Let $S \subset N$ with $1 \in S$. Suppose that for every $n \in S$ with $n > 1$ (at least) one of the following holds:*

(i) *For every integer m with $n < m$, $m \in S$.*

(ii) *For every integer m with $n \,\Delta\, m$, $m \in S$.*

Then there is a map $f \in C^0(S^1, S^1)$ such that $P(f) = S$.

PROOF. Let $S \subset N$ which satisfies the hypothesis. Suppose that, for all $n \in S$, $\{k \in N\colon n < k\}$ is not a subset of S. Then for all $n \in S$, $k \in S$ for every integer k with $n \,\Delta\, k$. By the theorem of Šarkovskii, there is a map $g \in C^0(I, I)$ such that $P(g) = S$. Hence, we can extend g to a map $f \in C^0(S^1, S^1)$ with $P(f) = S$.

Thus, we may assume that, for some $n \in S$, $\{k \in N\colon n < k\} \subset S$. We may choose n such that $\{k \in N\colon n \lessdot k\} \subset S$ but if $m < n$, $\{k \in N\colon m \lessdot k\}$ is not a subset of S. If $n = 1$ then $S = N$ and there are maps $f \in C^0(S^1, S^1)$ with $P(f) = N$. Hence we may assume that $n > 1$. Since $1 \in S$, this implies $n \gtrdot 3$.

Let $p_1, p_2, \ldots, p_n$ be distinct points on S^1 such that if $P = \{p_1, p_2, \ldots, p_n\}$ then $(p_i, p_{i+1}) \cap P = \varnothing$ for $i = 1, \ldots, n - 1$ and $(p_n, p_1) \cap P = \varnothing$. Let $e_2 \in (p_n, p_1)$ and let $e_1 \in (p_n, e_2)$.

We construct $f \in C^0(S^1, S^1)$ as follows. Let $f(p_i) = p_{i+1}$ for $i = 1, \ldots, n - 1$ and $f(p_n) = p_1$. Let $f(e_1) = e_1$ and $f(e_2) = e_2$. For $i = 1, \ldots, n - 2$, let f map the interval $[p_i, p_{i+1}]$ homeomorphically onto $[p_{i+1}, p_{i+2}]$. Let f map $[p_{n-1}, p_n]$ homeomorphically onto $[p_n, p_1]$. Also, let f map $[p_n, e_1]$ homeomorphically onto $[e_1, p_1]$ and let f map $[e_2, p_1]$ homeomorphically onto $[e_2, p_2]$.

It remains to define f on $[e_1, e_2]$. Let $T = \{i \in S\colon i < n\}$. Note that $T \neq \varnothing$ since $1 \in T$. There is a unique element k of T such that, for all $i \in T$ with $i \neq k$, $k \,\Delta\, i$. By Lemma 4, there is a map $g \in C^0([e_1, e_2], [e_1, e_2])$ with $g(e_1) = e_1$, $g(e_2) = e_2$ and $P(g) = \{k\} \cup \{j \in N\colon k \,\Delta\, j\}$. Define f on $[e_1, e_2]$ by $f(x) = g(x)$ for $x \in [e_1, e_2]$. Thus we have constructed $f \in C^0(S^1, S^1)$.

By construction e_1 and e_2 are fixed points of f and $\{p_1, p_2, \ldots, p_n\}$ is a periodic orbit of period n. It follows from Theorem A_1 that $m \in P(f)$ for every integer m with $m \gtrdot n$. Also, by construction, all periodic points outside the interval $[e_1, e_2]$ have period at least n.

Thus $P(f) = \{m \in N\colon n \lessdot m\} \cup \{k\} \cup \{m \in N\colon k \,\Delta\, m\} = S$. Q.E.D.

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