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ON THE WEIGHT AND PSEUDOWEIGHT OF LINEARLY ORDERED TOPOLOGICAL SPACES

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ABSTRACT. We derive a simple formula for the weight of a LOTS using the pseudoweight. As an application we give a very short proof of the nonorderability of the Sorgenfrey-line.

1. Definitions. Let (X, τ) be a T_1-space.

A collection $\mathfrak{U} \in \tau$ is called a ψ-base for X [3] if

(i) $\mathfrak{U}$ covers X, and

(ii) $\bigcap\{U | x \in U \in \mathfrak{U}\} = \{x\}$, for all $x \in X$.

We put as usual $\psi w(X) = \min\{|\mathfrak{U}| \,|\mathfrak{U}$ is a ψ-base for $X\}$.

Recall that

$$c(X) = \sup\{|\mathfrak{U}| \,|\mathfrak{U} \subset \tau \text{ and } \mathfrak{U} \text{ is disjoint}\},$$

and

$$w(X) = \min\{|\mathfrak{B}| \,|\mathfrak{B} \subset \tau \text{ and } \mathfrak{B} \text{ is a base for } X\}.$$

2.

THEOREM. *If X is a Linearly Ordered Topological Space (LOTS), then* $w(X) = c(X) \cdot \psi w(X)$.

PROOF. "$\geqslant$" is obvious.

"$\leqslant$". Let $\mathfrak{U} = \{U_i\}_{i \in I}$ be a ψ-base for X with $|\mathfrak{U}| = \psi w(X)$. For each $i \in I$ let $\{C_{i,j}\}_{j \in J_i}$ be the collection of convex components of U_i. Put $\mathfrak{B} = \{C_{i,j} | j \in J_i, i \in I\}$. Since $|J_i| \leqslant c(X)$ for all i, we see that $|\mathfrak{B}| \leqslant c(X) \cdot \psi w(X)$. We claim that $\mathfrak{B}$ is a subbase for X.

Indeed, take $x \in X$ and $(a, b) \ni x$. Since $\bigcap\{B | x \in B \in \mathfrak{B}\} = \{x\}$, there exist $B_a, B_b \in \mathfrak{B}$ such that $x \in B_a \not\ni a$ and $x \in B_b \not\ni b$. Then $x \in B_a \cap B_b \subset (a, b)$. $\square$

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3. Examples. We shall show that our theorem cannot be improved.

3.1. The Sorgenfrey-line S shows that we cannot replace "X is a LOTS" by "X is a GO-space". Indeed $w(S) = 2^\omega$, $c(S) = \omega$ and $\psi w(S) = \omega$ [take all intervals with rational endpoints].

This shows once again that S is not a LOTS. For other, more involved, proofs, see for example [1] and [4].

3.2. The largest "natural" invariant below $c(X)$ (in the case of LOTS) is $l(X)$, the Lindelöf number of X. We shall see that we cannot replace c by l:

Let $A \subset \mathbf{R}$ be a subset of cardinality 2^ω with the property that for any closed set $C \subset \mathbf{R}$ with either $C \subset A$ or $C \subset \mathbf{R} \setminus A$ we have that C is countable.

Build a Michael-line $M(A)$ by isolating every $a \in A$. The resulting space is Lindelöf [6]. Now let X be the associated LOTS of the GO-space $M(A)$ [5].

$X = \{\langle x, n\rangle \in \mathbf{R} \times \mathbf{Z} | x \in \mathbf{R} \setminus A \Rightarrow n = 0\}$ endowed with the lexicographic order. The map $f\colon X \to M(A)$ defined by $f(\langle x, n\rangle) = x$ is a retraction with countable fibers; hence X is Lindelöf.

Let us put $U_q = \{x \in X | x < \langle q, 0\rangle\}$, $V_q = \{x \in X | x > \langle q, 0\rangle\}$ and $O_n = \{\langle a, n\rangle | a \in A\}$. Then $\mathcal{U} = \{U_q\}_{q\in\mathbf{Q}} \cup \{V_q\}_{q\in\mathbf{Q}} \cup \{O_n\}_{n\in\mathbf{Z}}$ is a countable ψ-base for X. Finally we have $w(X) = |A| = 2^\omega$.

3.3. The largest invariant–to our knowledge–below $\psi w(X)$ is $psw(X) = \min\{\operatorname{ord}(\mathcal{U})|\ \mathcal{U}$ is a ψ-base for $X\}$ [2]. If we let Z be a dense left-separated subspace of a connected Souslin-line, then Z is a LOTS with a point-countable base [7]. Thus we have $w(Z) = \omega_1 > \omega = c(Z) \cdot psw(Z)$; hence ψw cannot be replaced by psw.

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