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A RESULT RELATED TO A THEOREM BY PIANIGIANI

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ABSTRACT. Let $\tau: J \to J$ be a piecewise C^2 map, where J is an interval, satisfying $\inf|\tau'| > 1$. An upper bound for the number of independent absolutely continuous measures invariant under τ is presented.

Introduction. Let $J = [a, b]$ be an interval, $\mathfrak{B}$ the Lebesgue measurable subsets of J, and λ the Lebesgue measure on J. Let $\tau: J \to J$ be a piecewise C^2 transformation satisfying $\inf|\tau'(x)| > 1$ where the derivative exists. In **[1]** it is shown that τ admits an absolutely continuous invariant measure μ, i.e., $\mu(A) = \mu(\tau^{-1}(A))$ for all $A \in \mathfrak{B}$, and

$$\mu(A) = \int_A f\,d\lambda,$$

where we refer to f as the *density invariant under* τ. Clearly $f \geqslant 0$ and $f \in \mathfrak{L}_1$, the space of integrable functions on J.

Let $\mathfrak{F}_\tau$ denote the space of densities invariant under τ and $\{a_1, a_2, \ldots, a_k\}$ those points in J where τ' does not exist. The main result of [2] asserts that dim $\mathfrak{F}_\tau \leqslant k$. In fact it is very easy to establish a better bound. Let $a = b_0 < b_1 < \cdots < b_m < b_{m+1} = b$ be the partition of J such that τ is continuous and monotonic on each interval (b_{i-1}, b_i). Clearly $m \leqslant k$, and dim $\mathfrak{F}_\tau \leqslant m$. In the special case where τ is continuous on J, the total number of peaks and valleys in the graph of τ constitutes an upper bound for dim $\mathfrak{F}_\tau$.

In §3 of [3] a still better bound is established for dim $\mathfrak{F}_\tau$. Let $\{b_1, b_2, \ldots, b_m\}$ be the partition defined in the previous paragraph. For each $1 \leqslant j \leqslant m$, define the pair

$$\langle u_j, v_j \rangle = \langle \tau(b_j^-), \tau(b_j^+) \rangle,$$

where u_j is regarded as u_j^+ or u_j^- depending on whether $\tau(a_j - \varepsilon) > u_j$ or $\tau(a_j - \varepsilon) < u_j$.

Two pairs $\langle u_i, v_i \rangle$ and $\langle u_j, v_j \rangle$ are said to be dependent if they have one or both coordinates in common. Otherwise the pairs are independent. Let N_τ denote the maximal number of independent pairs. Then Theorem 2 of [3] asserts that dim $\mathfrak{F}_\tau \leqslant N_\tau$. In this note we suggest a modified definition of dependence and present a different bound for the number of absolutely continuous measures invariant under τ.

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2. Dependence of densities. Let $\tau: J \to J$ be piecewise C^2 satisfying $\inf|\tau'(x)| > 1$ and let $\mathfrak{D} = \{b_1, b_2, \ldots, b_m\}$ be the partition on which τ is piecewise continuous and monotonic. We shall say that b_i and b_j are *dependent* if

$$\tau(b_i - \varepsilon, b_i + \varepsilon) \cap \tau(b_j - \varepsilon, b_j + \varepsilon)$$

has positive measure for every $\varepsilon > 0$. This implies, but is not equivalent to

$$\langle \tau(b_i^-), \tau(b_i^+) \rangle \cap \langle \tau(b_j^-), \tau(b_j^+) \rangle \neq \varnothing.$$

This definition of dependence for a pair of discontinuities in $\mathfrak{D}$ is reflexive, symmetric, but not transitive. A collection $\mathbb{S} \subset \mathfrak{D}$ is said to be *dependent* if every pair of points in this collection is dependent, and maximal if $\mathbb{S}$ is not a proper subset of any dependent collection. Notice that two distinct maximal dependent collections may have nonempty intersection, and such a collection may consist of a single point. Thus, given $b_j \in \mathfrak{D}$, there exists at least one and at most two maximal dependent collections containing b_j. In particular, when τ is continuous at b_j, there exists only one maximal dependent collection containing this point. Let H_τ be the number of distinct maximal dependent collections. Then, we have

THEOREM. $\dim \mathfrak{F}_\tau \leqslant H_\tau$.

PROOF. We first show that if f_1 and f_2 are invariant with disjoint supports, then to each f_i there corresponds one maximal dependent collection S_i and $S_1 \neq S_2$. Letting $M_i = \operatorname{spt} f_i$, it is easy to see that int M_i has to contain at least one point of $\mathfrak{D}$, say b_i'. Let S_1 and S_2 be any maximal collections containing b_1' and b_2', respectively, and suppose $S_1 = S_2$. Then b_1' and b_2' are dependent. Since $\tau(M_i) \subset M_i$ a.e. [1], and $(b_i' - \varepsilon, b_i' + \varepsilon) \subset M_i$ for some $\varepsilon < 0$, the dependence of b_1' and b_2' implies

$$\lambda(M_1 \cap M_2) \geqslant \lambda\left[\tau(b_1' - \varepsilon, b_1' + \varepsilon) \cap \tau(b_2' - \varepsilon, b_2' + \varepsilon)\right] > 0.$$

This is a contradiction. Therefore, S_1 and S_2 must be distinct.

Now let $\{f_1, f_2, \ldots, f_n\}$ be a maximal set of disjoint densities invariant under τ [2]. By the preceding argument we see that there exists a 1-1 mapping from $\{f_1, \ldots, f_n\}$ into $\{S_1, \ldots, S_{H_\tau}\}$. Thus $n \leqslant H_\tau$. Q.E.D.

3. Examples. (a) Consider the transformation τ shown in Figure 1.

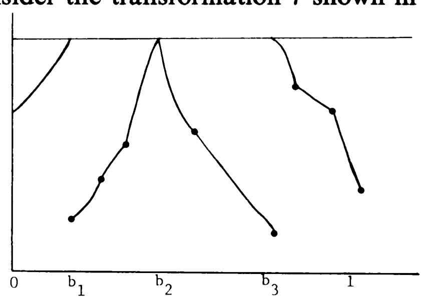

FIGURE 1

We see that $\{b_1, b_2, b_3\}$ is the unique collection which is dependent and maximal. Thus $H_\tau = 1$ and there exists a unique absolutely continuous measure invariant under τ. The bound from [2] is 8, since there are 8 discontinuities in τ' in $(0, 1)$.

(b) Let τ have the graph shown in Figure 2.

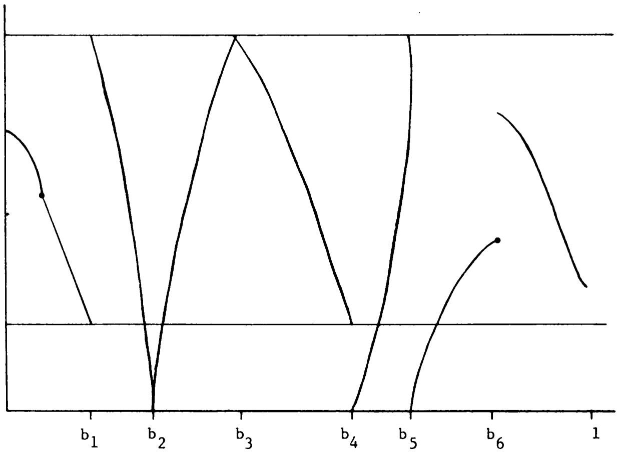

FIGURE 2

For each discontinuity, we give the corresponding maximal dependent collection or collections as the case may be:

b_1: $\{b_1, b_3, b_5\}$ and $\{b_1, b_4\}$,
b_2: $\{b_2, b_4, b_5\}$,
b_3: $\{b_1, b_3, b_5\}$,
b_4: $\{b_1, b_4\}$ and $\{b_2, b_4, b_5\}$,
b_5: $\{b_1, b_3, b_5\}$ and $\{b_2, b_4, b_5\}$,
b_6: $\{b_6\}$.

There are 4 independent collections. Therefore τ admits at most four independent invariant densities.

Notice that for this example the bound of [2] is 7, since there are 7 discontinuities of τ' in $(0, 1)$.

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